UNIVERSITY OF TWENTE

Department of Electrical Engineering, Mathematics and Computer Science

Solution exam Mathematics C1 Cayley on Friday March 31, 2017, 13.45 – 15.45 hours.

The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer!

You are **not** allowed to use a formula sheet and you are **not** allowed to use a calculator.

1.

Given is the following linear system of equations:

$$\begin{cases} 6x_1 + 2x_2 - 6x_4 = 6 \\ -3x_1 + x_2 + 2x_3 + x_4 = -3 \end{cases}$$

This system can also be written as $A_1\mathbf{x} = \mathbf{b}_1$ where A_1 is the coefficient matrix and $\mathbf{b}_1$ is the right-hand side of this system.

a) Determine the solution set of this system ans write it in parametric vector form.

The associated augmented matrix is given by:

$$\left(\begin{array}{cccc|c} 6 & 2 & 0 & -6 & 6 \\ -3 & 1 & 2 & 1 & -3 \end{array}\right)$$

Using elementary row operations we obtain:

$$\left(\begin{array}{cccc|c} 6 & 2 & 0 & -6 & 6 \\ -3 & 1 & 2 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 3 & 1 & 0 & -3 & 3 \\ -3 & 1 & 2 & 1 & -3 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 3 & 1 & 0 & -3 & 3 \\ 0 & 2 & 2 & -2 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 3 & 1 & 0 & -3 & 3 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 3 & 0 & -1 & -2 & 3 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

We obtain:

$$\begin{cases} x_1 & = 1 + \frac{1}{3}x_3 + \frac{2}{3}x_4 \\ x_2 & = -x_3 + x_4 \\ x_3 & \text{is free} \\ x_4 & \text{is free} \end{cases}$$

In parametric vector form this yields:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 3 \end{pmatrix} \right\}.$$

b) Determine Null A_1.

We obtain:

$$\text{Null}\, A_1 = \text{Span}\left\{\begin{pmatrix}1\\-3\\3\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\\3\end{pmatrix}\right\}.$$

Given is a second linear system of equations:

$$\begin{cases} -3x_1 + x_2 + 2x_3 + x_4 = -3 \\ \quad x_3 - x_4 = 0 \end{cases}$$

This second system can also be written as $A_2\mathbf{x} = \mathbf{b}_2$ where A_2 is the coefficient matrix and $\mathbf{b}_2$ is the right-hand side of this second system.

c) Determine all vectors $\mathbf{x}$ which satisfy $A_1\mathbf{x} = \mathbf{b}_1$ as well as $A_2\mathbf{x} = \mathbf{b}_2$.

We can simply combine the two linear systems. Two steps can simplify this. First, note that the second equation of the first system is the same as the first equation of the second system. We can simply delete that equation. Second, we can start with the reduced echelon form of the first system. Hence we need to solve a linear system with augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

Using elementary row operations we obtain:

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

We obtain:

$$\begin{cases} x_1 = 1 + x_4 \\ x_2 = 0 \\ x_3 = x_4 \\ x_4 \text{ is free} \end{cases}$$

In parametric vector form this yields:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \text{Span}\left\{\begin{pmatrix}1\\0\\1\\1\end{pmatrix}\right\}.$$

2\. The matrices A and B are given by:

$$A = \begin{pmatrix} \alpha & 4-2\alpha & \alpha-1 \\ \alpha & 2-\alpha & \alpha-1 \\ -2\alpha & 2-\alpha & 1-\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 & 2 \end{pmatrix}$$

a) Determine all $\alpha \in \mathbb{R}$ for which A is invertible.

Using elementary row operations, we have:

$$\begin{pmatrix} \alpha & 4-2\alpha & \alpha-1 \\ \alpha & 2-\alpha & \alpha-1 \\ -2\alpha & 2-\alpha & 1-\alpha \end{pmatrix} \sim \begin{pmatrix} \alpha & 4-2\alpha & \alpha-1 \\ 0 & -2+\alpha & 0 \\ 0 & 10-5\alpha & \alpha-1 \end{pmatrix}$$
$$\sim \begin{pmatrix} \alpha & 0 & \alpha-1 \\ 0 & -2+\alpha & 0 \\ 0 & 0 & \alpha-1 \end{pmatrix}$$
$$\sim \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -2+\alpha & 0 \\ 0 & 0 & \alpha-1 \end{pmatrix}$$

(or use the determinant). It is then clear that the matrix is invertible if and only if $\alpha \neq \{0, 1, 2\}$.

b) Take $\alpha = -1$. Determine A^{-1}.

We have:

$$A = \begin{pmatrix} -1 & 6 & -2 \\ -1 & 3 & -2 \\ 2 & 3 & 2 \end{pmatrix}$$

For computation of the inverse, we consider the following expanded matrix:

$$\left(A \mid I \right) = \left(\begin{array}{ccc|ccc} -1 & 6 & -2 & 1 & 0 & 0 \\ -1 & 3 & -2 & 0 & 1 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & -6 & 2 & -1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 1 & 0 \\ 0 & 15 & -2 & 2 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -2 & 0 \\ 0 & -3 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & -3 & 5 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & -3 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & -3 & 5 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \end{array}\right)$$

and hence

$$A^{-1} = \begin{pmatrix} -2 & 3 & 1 \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \end{pmatrix}$$

c) Take $\alpha = -1$. Determine a matrix C such that $CA = B$.

Since we know that A is invertible, we have:

$$C = BA^{-1}$$

We find:

$$C = \begin{pmatrix} 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 & 1 \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

3.

Given are three vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ such that:

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3\}$$

Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

We clearly have

$$\mathbf{v}_1 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3\}$$

This implies there exists α and β such that:

$$\mathbf{v}_1 = \alpha\mathbf{v}_2 + \beta\mathbf{v}_3$$

Since one of the vectors, namely $\mathbf{v}_1$ in $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linear combination of the other vectors in S we find, by defintion, that the set of vectors in S is linearly dependent.

4.

The matrix A is given by:

$$A = \begin{pmatrix} 0 & -5 & -3 \\ -2 & 0 & 0 \\ 2 & -4 & -4 \end{pmatrix}$$

a) Show that 2 is an eigenvalue of the matrix A.

Using elementary row operations we obtain:

$$2I - A = \begin{pmatrix} 2 & 5 & 3 \\ 2 & 2 & 0 \\ -2 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 2 & 5 & 3 \\ 0 & -3 & -3 \\ 0 & 9 & 9 \end{pmatrix} \sim \begin{pmatrix} 2 & 5 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly $2I - A$ is not invertible. This implies that 2 is an eigenvalue of the matrix A. Moreover, we immediately obtain the associated eigenspace

$$E_2 = \text{Null}(2I - A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

b) Determine the eigenvalues of A and the corresponding eigenspaces.

Using elementary row operations we obtain:

$$\lambda I - A = \begin{pmatrix} \lambda & 5 & 3 \\ 2 & \lambda & 0 \\ -2 & 4 & \lambda+4 \end{pmatrix} \sim \begin{pmatrix} \lambda & 5 & 3 \\ 2 & \lambda & 0 \\ 0 & \lambda+4 & \lambda+4 \end{pmatrix}$$

We obtain:

$$\det(\lambda I - A) = (\lambda+4)\det\begin{pmatrix} \lambda & 5 & 3 \\ 2 & \lambda & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Again, using elementary row operations we obtain:

$$\begin{pmatrix} \lambda & 5 & 3 \\ 2 & \lambda & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda & 0 & -2 \\ 2 & 0 & -\lambda \\ 0 & 1 & 1 \end{pmatrix}$$

Therefore (expanding around second column):

$$\det(\lambda I - A) = (\lambda+4)\det\begin{pmatrix} \lambda & 0 & -2 \\ 2 & 0 & -\lambda \\ 0 & 1 & 1 \end{pmatrix} = (\lambda+4)(\lambda^2-4)$$

In other words, we have eigenvalues $2, -2$ and -4.

We already found the eigenspace associated with eigenvalue 2. Remains the eigenspaces associated with eigenvalues -4 and -2.

Using elementary row operations we obtain:

$$-2I - A = \begin{pmatrix} -2 & 5 & 3 \\ 2 & -2 & 0 \\ -2 & 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We immediately obtain the associated eigenspace

$$E_{-2} = \text{Null}(-2I - A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$$

Using elementary row operations we obtain:

$$-4I - A = \begin{pmatrix} -4 & 5 & 3 \\ 2 & -4 & 0 \\ -2 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} -4 & 5 & 3 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -3 & 3 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We immediately obtain the associated eigenspace

$$E_{-4} = \text{Null}(-4I - A) = \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right\}$$

c) Determine the eigenvalues of A^5 and the corresponding eigenspaces.

If $A\mathbf{x} = \lambda\mathbf{x}$ then we obtain $A^5\mathbf{x} = \lambda^5\mathbf{x}$. In other words, A^5 has eigenvalues $2^5 = 32$, $(-2)^5 = -32$ and $(-4)^5 = -1024$ with associated eigenspaces:

$$E_{32} = \text{Span}\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}, \qquad E_{-32} = \text{Span}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}, \qquad E_{-1024} = \text{Span}\left\{\begin{pmatrix}2\\1\\1\end{pmatrix}\right\}$$

5.

Given is that the eigenvalues of the matrix $A \in \mathbb{R}^{3\times3}$ are given by 1, 0 and -1 with eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, respectively:

$$\mathbf{p}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \mathbf{p}_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad \mathbf{p}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

Determine the matrix A.

We have $A = PDP^{-1}$ with P a matrix whose columns are the eigenvectors of A and D a diagonal matrix whose diagonal entries are the eigenvalues of A. We have:

$$P = \begin{pmatrix}0&1&1\\1&1&1\\0&1&0\end{pmatrix}, \qquad D = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

and

$$\left(P \mid I \right) = \left(\begin{array}{ccc|ccc}0&1&1&1&0&0\\1&1&1&0&1&0\\0&1&0&0&0&1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1&1&1&0&1&0\\0&1&1&1&0&0\\0&1&0&0&0&1\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1&0&0&-1&1&0\\0&1&1&1&0&0\\0&0&-1&-1&0&1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1&0&0&-1&1&0\\0&1&1&1&0&0\\0&0&1&1&0&-1\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1&0&0&-1&1&0\\0&1&0&0&0&1\\0&0&1&1&0&-1\end{array}\right)$$

and

$$P^{-1} = \begin{pmatrix}-1&1&0\\0&0&1\\1&0&-1\end{pmatrix}$$

but then

$$A = PDP^{-1} = \begin{pmatrix}0&1&1\\1&1&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}-1&1&0\\0&0&1\\1&0&-1\end{pmatrix} = \begin{pmatrix}0&1&1\\1&1&1\\0&1&0\end{pmatrix}\begin{pmatrix}-1&1&0\\0&0&0\\-1&0&1\end{pmatrix}$$

$$= \begin{pmatrix}-1&0&1\\-2&1&1\\0&0&0\end{pmatrix}$$

6.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which first rotates each point $(x_1, x_2) \in \mathbb{R}^2$ over 45 degrees (counter clockwise) and then scales the result by $\sqrt{2}$.

a) Determine the representation matrix of T.

We compute the image of the standard basis vectors. We get

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix} \to \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

where in each case I first rotate and then scale. I find:

$$[T] = A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

b) Determine the representation matrix of T^{16}.

We can note that when rotating and scaling the order does not affect the result. Therefore, we need to rotate in total over $16 \cdot 45 = 720$ degrees which does not do anything and scale by $\sqrt{2}^{16} = 2^8 = 256$. Therefore

$$[T^{16}] = \begin{pmatrix} 256 & 0 \\ 0 & 256 \end{pmatrix}$$

Alternatively, you can use $[T^{16}] = [T]^{16}$

$$[T]^2 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$[T]^4 = [T]^2[T]^2 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$$

$$[T]^8 = [T]^4[T]^4 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$$

$$[T]^{16} = [T]^8[T]^8 = \begin{pmatrix} 256 & 0 \\ 0 & 256 \end{pmatrix}$$

7.

Given are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$:

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

a) Determine a basis $\mathcal{B}$ for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

We have using elementary row operations:

$$\begin{pmatrix} 3 & -2 & 1 & 0 \\ 6 & -4 & 1 & 1 \\ -3 & 2 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see pivots in column 1 and 3. Hence a basis is

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_3\}$$

but there are many other options.

b) Determine $[\mathbf{v}_5]_{\mathcal{B}}$ for $\mathbf{v}_5 = \mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4$.

We have:

$$\mathbf{v}_5 = \begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix} = \tfrac{4}{3}\mathbf{v}_1$$

Hence

$$[\mathbf{v}_5]_{\mathcal{B}} = \begin{pmatrix} \frac{4}{3} \\ 0 \end{pmatrix}$$

8.

Given are the matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Determine

$$\det(B^{\mathsf{T}} A^{-1} B A)$$

We have:

$$\begin{aligned} \det(B^{\mathsf{T}} A^{-1} B A) &= \det(B^{\mathsf{T}}) \det(A^{-1}) \det(B) \det(A) \\ &= \det(B) \det(A^{-1}) \det(B) \det(A) \\ &= \det(B) \det(A)^{-1} \det(B) \det(A) \\ &= \det(B)^2 \end{aligned}$$

Since B is upper diagonal we easily see that $\det(B) = 6$ and hence:

$$\det(B^{\mathsf{T}} A^{-1} B A) = 36$$