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LINEAR ALGEBRA$_D$ Date : April 21, 2023
Time : 13.45 – 15.45 hrs

First read these instructions carefully:

This test consists of 10 exercises.
For exercises 1, 4, and 6 you must only write down the final answers on the answer sheet. The complete solutions for the remaining exercises must be accurately written down on the answer sheet, including calculations and argumentation. The use of dictionaries or electronic devices is not allowed. <u>The answer sheet must be handed in.</u> The grade is calculated by dividing the total number of points by 4, then adding 1.

1. (3pts.) *This is a final answer question. Provide your final answer (and only your final answer) on the supplied space on the answer sheet.*

   The matrix given is the augmented matrix for a system of linear equations. Give the parametric vector form for the general solution of this system of linear equations.

$$\left(\begin{array}{ccccc|c} 1 & 0 & 5 & 0 & -1 & 0 \\ 0 & 1 & -5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

2. (3pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   Let $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

   Calculate the following expression: $AB^T + \mathbf{v}\mathbf{v}^T$

3. (3pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   Given $N = \begin{pmatrix} 7 & 1 \\ 2 & 8 \end{pmatrix}$. Find a regular (invertible) 2×2 matrix M such that $M^3 = M^2N - 5M^2$.

4. (3pts.) *This is a final answer question. Provide your final answer (and only your final answer) on the supplied space on the answer sheet.*

   Determine which of the following set(s) form a basis of $\mathbb{R}^3$. *Only one option is correct.*

   (A) $S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix} \right\}$

   (B) $S_2 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$

   (C) $S_3 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \right\}$

   (D) $S_4 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$

5. (4pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   Let $C \in \mathbb{R}^{3\times 3}$ and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be linearly independent vectors in $\mathbb{R}^3$.

   We are given: $C\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $C\mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $C\mathbf{z} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

   Find the value of the determinant of the matrix C.

6. *This is a final answer question. Provide your final answer (and only your final answer) on the supplied space on the answer sheet.*

   The matrix D is given as $D = \begin{pmatrix} -1 & -5 \\ 5 & -1 \end{pmatrix}$.

   a) (2pts.) Determine the eigenvalues of D.

   b) (2pts.) Determine the corresponding eigenspaces of D.

7. *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   Consider two planes p_1 and p_2 in $\mathbb{R}^3$ given as below.

   p_1: $\quad 2x_1 + 3x_2 - 2x_3 + 2 = 0$

   p_2: $\quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

   a) (1pt.) Provide an equation for p_2 similar to the equation for p_1.

   b) (3pts.) Find the intersection of these two planes.

8. *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be a linear transformation given with:

$$T(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_1 + x_2 - 3x_3 \\ x_1 - 3x_2 - 2x_3 \\ x_1 + 2x_2 - 5x_3 \end{pmatrix}$$

   a) (2pts.) Let A be the associated representation matrix for T, determine A.

   b) (3pts.) Is T surjective (onto) and/or injective (one-to-one)? Explain your answer, providing all relevant arguments.

9. (3pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   Consider the linear transformation P which rotates every vector in $\mathbb{R}^2$ about the origin through an angle of $\frac{\pi}{4}$ radians (counterclockwise).

   Determine the representation matrix of P by examining the images of $\mathbf{e}_1$ and $\mathbf{e}_2$.

10. (4pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

    Let A and $B \in \mathbb{R}^{n \times n}$. Suppose that the matrices A and B have a common eigenvector which is $\mathbf{v}$. Show that $\det(AB - BA) = 0$.

6. (2pts.) *This is a final answer question. Provide your final answer (and only your final answer) on the supplied space on the answer sheet.*

   Find the volume of the parallelepiped in $\mathbb{R}^3$ which has corners: $\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ given as:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}.$$

7. (3pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   The matrix A is given by:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   Show that if $ad - bc \neq 0$ then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

8. (3pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   Let a matrix A has eigenvalues: -1, 0, and 1. Determine all of the eigenvalues of $5I + 3A$.

9. (4pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   The 3×3 matrix N is given:

$$N = \begin{pmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{pmatrix}$$

   Determine the matrices P and D, such that $N = PDP^{-1}$.

10. *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

   Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that:

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}.$$

   a) (3pts.) Find a general formula for $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

   b) (3pts.) Determine whether T is surjective (onto) and/or injective (one-to-one).