Resit Test for Mathematics C (Cayley) for
20 April 2018 13:45-15:45

The solutions to the questions need to be well structured and clearly formulated.
All answers should be motivated.
The use of electronic devices is not allowed.

1. Consider the following system of linear equations

$$\begin{cases} 6x_1 + x_2 - x_3 + 12x_4 = 4, \\ -x_1 + x_2 + x_3 - 3x_4 = 3, \\ -3x_1 \qquad + x_3 - 6x_4 = 0. \end{cases}$$

A is the coefficient matrix of the system.
a) [3 pt] Determine the solution set of the system and write it in parametric vector form. Indicate in each step which elementary row operation you use.
b) [3 pt] Clearly state the definition for the null space of a general $m \times n$ matrix B. Determine Null A for the given system.
c) [2 pt] Is the system $A\mathbf{x}=\mathbf{b}$ consistent for all $\mathbf{b} \in \mathbb{R}^3$?

2. [3 pt] Use determinants to evaluate the area of the parallelogram with vertices $(1,1)$, $(4,2)$, $(5,4)$ and $(2,3)$.

3. The matrices A and B are given by

$$A = \begin{pmatrix} \alpha^2 & 3\alpha \\ 2\alpha+1 & 10 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 2 & 3 \\ 1 & -1 & 1 \end{pmatrix}.$$

a) [2 pt] Determine all $\alpha \in \mathbb{R}$ for which A is not invertible.
b) [2 pt] Let $\alpha = 1$. Determine A^{-1}.
c) [2 pt] Let $\alpha = 1$. Determine a matrix C such that $AC = B$.

4. The matrix A is given by

$$A = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}.$$

a) [4 pt] Determine the (real or complex) eigenvalues and associated eigenspaces of A.
b) [2 pt] A is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix. Suggest a suitable P and D which will satisfy this expression.

5. The matrix A and the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are given by

$$A = \begin{pmatrix} 1 & 2 & -7 \\ 0 & 1 & -3 \\ 0 & -3 & 9 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix} \text{ and } \mathbf{b}_2 = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$$

a) [2 pt] Show that $\mathbf{b}_1 \in \mathrm{Col}\, A$ and that $\mathbf{b}_2 \notin \mathrm{Col}\, A$.
b) [2 pt] Determine a basis $\mathcal{B}$ for Col A and determine $[\mathbf{b}_1]_{\mathcal{B}}$.

6. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that first reflects each point $(x_1, x_2) \in \mathbb{R}^2$ across the line $x_1 = -x_2$ followed by *clockwise* rotation through $\frac{\pi}{4}$.
a) [3 pt] Determine the representation matrix of T.
b) [1 pt] Is T an invertible transformation? Motivate your answer.
c) [2 pt] Determine if T is onto and if T is one-to-one.

7. [3 pt] Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Show that $(A^{-1})^T = (A^T)^{-1}$.

Total: 36 points. Grade $= 1 + \frac{\text{Score}}{4}$

$$5\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + 1\begin{bmatrix} -7 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix}$$