UNIVERSITY OF TWENTE

Department of Electrical Engineering, Mathematics and Computer Science

Solution exam Linear Algebra on Friday April 17, 2020, 13.45 – 15.45 hours.

2.

> Given are two matrices:
>
> $$A = \begin{pmatrix} 1 & 1 & \alpha \\ 1 & \beta & 2 \\ 1 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$
>
> It is known that B is an echelon form of the matrix A. Determine α and β.

There are two ways to determine α and β. First we have $\operatorname{Null} A = \operatorname{Null} B$ and

$$\operatorname{Null} B = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Hence

$$\begin{pmatrix} 1 & 1 & \alpha \\ 1 & \beta & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \mathbf{0}$$

which yields $\alpha = 3$ and $\beta = \frac{1}{2}$.

Alternatively:

$$\begin{aligned} A = \begin{pmatrix} 1 & 1 & \alpha \\ 1 & \beta & 2 \\ 1 & 0 & 1 \end{pmatrix} &\sim \begin{pmatrix} 1 & 1 & \alpha \\ 0 & \beta - 1 & 2 - \alpha \\ 0 & -1 & 1 - \alpha \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & \alpha \\ 0 & 1 & \alpha - 1 \\ 0 & \beta - 1 & 2 - \alpha \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & \alpha \\ 0 & 1 & \alpha - 1 \\ 0 & \beta - 1 & 2 - \alpha \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \alpha - 1 \\ 0 & 0 & 2 - \alpha - (\beta - 1)(\alpha - 1) \end{pmatrix} \end{aligned}$$

Since this should be similar to B we find $2 - \alpha - (\beta - 1)(\alpha - 1) = 0$ and $\alpha = 3$. This then yields $\alpha = 3$ and $\beta = \frac{1}{2}$.

3.

> We know
>
> $$\begin{pmatrix} 1 \\ 3 \\ \alpha \end{pmatrix} \in \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\}$$
>
> Determine α.

We should have:

$$\lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ \alpha \end{pmatrix}$$

The first two elements yield $\lambda - 2\mu = 1$ and $\lambda + \mu = 3$. This given $\lambda = \frac{7}{3}$ and $\mu = \frac{2}{3}$. The final element then yields:

$$-\lambda + 2\mu = \alpha \quad \implies \quad \alpha = -1$$

4.

Given is that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis of the subspace $\mathcal{V}$. Show that $\{\mathbf{x}_1, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3\}$ is also a basis of the subspace $\mathcal{V}$.

There are two issues: independence and spanning $\mathcal{V}$. We note that

$$\alpha \mathbf{x}_1 + \beta(\mathbf{x}_1 + \mathbf{x}_2) + \gamma(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) = \mathbf{0}$$

yields:

$$(\alpha + \beta + \gamma)\mathbf{x}_1 + (\beta + \gamma)\mathbf{x}_2 + \gamma \mathbf{x}_3 = \mathbf{0}$$

and independence of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ yields

$$\alpha + \beta + \gamma = 0, \quad \beta + \gamma = 0, \quad \gamma = 0$$

which yields $\alpha = \beta = \gamma = 0$ and hence we have independence.

Next we note

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_1 \\ \mathbf{x}_2 &= -\mathbf{x}_1 + (\mathbf{x}_1 + \mathbf{x}_2) \\ \mathbf{x}_3 &= -(\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) \end{aligned}$$

which proves $\mathcal{V} \subset \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. Trivially, we have $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \subset \mathcal{V}$ and hence the argument is complete.

5.

Given are matrices A and B: $$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$$ Find all matrices X such that $AX = B$.

If we choose:

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

then $AX = B$ yields:

$$\begin{aligned} x_1 - x_3 &= -2 \\ x_2 - x_4 &= 1 \\ -x_1 + x_3 &= 2 \\ -x_2 + x_4 &= -1 \end{aligned}$$

The third and fourth equation are the same as the first two equations and hence we find:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

or

$$X = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} + \operatorname{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

6.

Given is the matrix

$$A = \begin{pmatrix} 1 & 2 & \alpha \\ 1 & \alpha & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

with $\alpha \in \mathbb{R}$.

a) Find all $\alpha \in \mathbb{R}$ for which

$$\operatorname{Null} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

We must have

$$A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{0}$$

which yields $\alpha = 1$. Strictly speaking we still need to show that $\operatorname{Null} A$ is not larger than the given span but this is obvious since for $\alpha = 1$:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b) Find all $\alpha \in \mathbb{R}$ for which

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We basically need to show that $\operatorname{Col} A = \mathbb{R}^3$. We have:

$$A = \begin{pmatrix} 1 & 2 & \alpha \\ 1 & \alpha & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & \alpha \\ 0 & \alpha - 2 & 1-\alpha \\ 0 & -1 & 1-\alpha \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 2 & \alpha \\ 0 & -1 & 1-\alpha \\ 0 & \alpha - 2 & 1-\alpha \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 2 & \alpha \\ 0 & 1 & \alpha - 1 \\ 0 & \alpha - 2 & 1-\alpha \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 2 & \alpha \\ 0 & 1 & \alpha - 1 \\ 0 & 0 & 1-\alpha-(\alpha-2)(\alpha-1) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & \alpha \\ 0 & 1 & \alpha - 1 \\ 0 & 0 & -(\alpha-1)^2 \end{pmatrix}$$

which shows that $\operatorname{Col} A = \mathbb{R}^3$ if $\alpha \neq 1$.

7.

Given is that for the matrix

$$A = \begin{pmatrix} 3 & -2 & -2 \\ -1 & 1 & 2 \\ -4 & 3 & \alpha \end{pmatrix}$$

the inverse is given by:

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 8 & -2 \\ -1 & 13 & -4 \\ 1 & -1 & 1 \end{pmatrix}$$

Determine α.

We must have

$$I = A^{-1}A = \frac{1}{3} \begin{pmatrix} 1 & 8 & -2 \\ -1 & 13 & -4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 & -2 \\ -1 & 1 & 2 \\ -4 & 3 & \alpha \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 14-2\alpha \\ 0 & 3 & 28-4\alpha \\ 0 & 0 & -4+\alpha \end{pmatrix}$$

which yields $\alpha = 7$.

8.

Given is the matrix

$$A = \begin{pmatrix} 5 & -1 & 3 & -1 \\ 3 & 1 & 3 & -1 \\ 2 & 2 & 3 & -1 \\ -3 & -4 & 1 & 8 \end{pmatrix}$$

Find all α for which

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \alpha \end{pmatrix}$$

is an eigenvector of the matrix A.

We must have:

$$\begin{pmatrix} 5 & -1 & 3 & -1 \\ 3 & 1 & 3 & -1 \\ 2 & 2 & 3 & -1 \\ -3 & -4 & 1 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \alpha \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ \alpha \end{pmatrix}$$

which yields:

$$\begin{pmatrix} 7-\alpha \\ 7-\alpha \\ 7-\alpha \\ -6+8\alpha \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ \alpha \end{pmatrix}$$

The first three elements yield $\lambda = 7 - \alpha$. The final element yields:

$$-6 + 8\alpha = (7 - \alpha)\alpha$$

or

$$\alpha^2 + \alpha - 6 = 0$$

This gives $\alpha = -3$ or $\alpha = 2$.

9.

Given is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad T^2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

a) Determine the representation matrix of T.

Substituting the first in the second equation we find:

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

We have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\tfrac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

but then

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence the representation matrix is:

$$\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$$

b) Determine whether the transformation T is onto.

We clearly have

$$\operatorname{Im} T = \operatorname{Span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

and hence T is **not** onto.