Solution/Correction standard, Test Mathematics A; September 19, 2014.

1. $A_4 = \left\{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\right\} = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and $A_6 = \left\{\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}\right\} = \left\{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$.

   So: $A_4 \cap A_6 = \left\{\frac{1}{2}, 1\right\}$; **[1 pt]**

   $A_4 \cup A_6 = \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, 1\right\}$ (not necessarily in this order); **[1 pt]**

   $A_4 - A_6 = \left\{\frac{1}{4}, \frac{3}{4}\right\}$. **[1 pt]**

   Furthermore, $\bigcap_{k=2}^{10} A_k = \{1\}$. **[1 pt]**

2. A quantified statement for $\overline{A} \subseteq B$ is:

   $\forall x\,(x \notin A \to x \in B)$ or $\forall x\,(\neg(x \in A) \to x \in B)$. **[2 pt]**
   A statement that is logically not entirely correct: **[0 pt]**

3. (a) Distinguish the cases $|x| \geq |y|$ and $|x| < |y|$.

   If $|x| \geq |y|$, then $||x| - |y|| = |x| - |y| \leq |x| + |y|$, since $|y| \geq 0$. **[1 pt]**

   If $|x| < |y|$, then $||x| - |y|| = -\left(|x| - |y|\right) = -|x| + |y| \leq |x| + |y|$, since $|x| \geq 0$. **[1 pt]**

   (b) <u>Basis step</u> for $n = 1$:

   $$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot (1+1)} = \frac{1}{2} \quad \text{and also} \quad \frac{1}{1+1} = \frac{1}{2}.$$
   So the statement is correct for $n = 1$. **[0.5 pt]**

   <u>Induction step</u>:

   Let $k \geq 1$ and suppose that:
   $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$. (Induction hypothesis: IH). **[0.5 pt]**

   We must show that IH implies: $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(k+1)+1}$. **[0.5 pt]**

   Well: $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)((k+1)+1)}$.

   By IH this expression is equal to $\frac{k}{k+1} + \frac{1}{(k+1)((k+1)+1)}$. **[0.5 pt]**
   Now it remains to show that
   $\frac{k}{k+1} + \frac{1}{(k+1)((k+1)+1)} = \frac{k+1}{(k+1)+1}$. **[0.5 pt]**

This is straightforward:
$$\frac{k}{k+1}+\frac{1}{(k+1)((k+1)+1)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)},$$
and also $\dfrac{k+1}{(k+1)+1}=\dfrac{k+1}{k+2}=\dfrac{(k+1)^2}{(k+1)(k+2)}$. **[0.5 pt]**

Now we obtain from the principle of mathematical induction that for all $n \geq 1$:
$$\sum_{i=1}^{n}\frac{1}{i(i+1)}=\frac{n}{n+1}.$$
(From the proof it must be crystal clear whàt is supposed and whàt must be proved. In case of nonsense formulations like "Suppose it is correct FOR ALL n, so it also holds for $n+1$": at most 1 pt for the entire exercise)

4. (a) Since the order in which the cards are selected matters (card for the first child, card for the second child, etc), we need to determine the number of 13-permutations of a set of 52 elements, which is: $P(52,13)=\dfrac{52!}{(52-13)!}=52\cdot 51\cdot\dots\cdot 40.$ **[1 pt]**

   Answer **[0.5 pt]**; argumentation **[0.5 pt]**.

   Alternatively: There are 52 choices for the first child. Once this child is given a card, there are 51 choices left for the second child. Subsequently, there are 50 choices for the third child, etc, and finally, 40 choices for the last child.
   So by the rule of product, the number of possibilities is $52\cdot 51\cdot\dots\cdot 40$.

   (b) There are $\binom{13}{5}$ possibilities to choose 5 hearts from 13 (order does not matter) and also $\binom{13}{5}$ possibilities to choose 5 spades from 13. The remaining 3 cards must be chosen from the 26 clubs and diamonds; which can be done in $\binom{26}{3}$ ways. **[1 pt]**

   So, by the rule of product, there are $\binom{13}{5}\cdot\binom{13}{5}\cdot\binom{26}{3}$ possibilities. **[1 pt]**

   Answer **[1 pt]**; argumentation **[1 pt]**.