Exam 2, Module 7, Code 201700304

Discrete Structures & Efficient Algorithms

Friday April 6, 2018, 13:45 - 16:45

All answers need to be motivated. No calculators. You are allowed to use a handwritten cheat sheet (A4) per topic (L&M,ALG,DM). Also if you cannot solve a part of a question you may use that result in subsequent parts of the question.

This exam consists of three parts, with the following (estimated) times:

Languages & Machines (L&M)	1h	(30 points)
Algebra (ALG)	1h 40 min	(50 points)
Discrete Mathematics (DM)	20 min	(10 points)

Total of 30+50+10=90 points. Including 10 bonus points that makes 100 points. Your exam grade is the total number pf points divided by 10.

Please use a new sheet of paper for each part (L&M/ALG/DW)!

Languages & Machines

1. (5 points) Which variables can be derived by chain rules from C in the following grammar G_1? Provide an equivalent grammar without chain rules.

$$G_1 = \begin{cases} S & \to & AB \mid C \\ A & \to & aA \mid B \\ B & \to & bB \mid C \\ C & \to & cC \mid a \mid A \end{cases}$$

2. (5 points) Provide a *regular* grammar, equivalent to the following context-free grammar G_2:

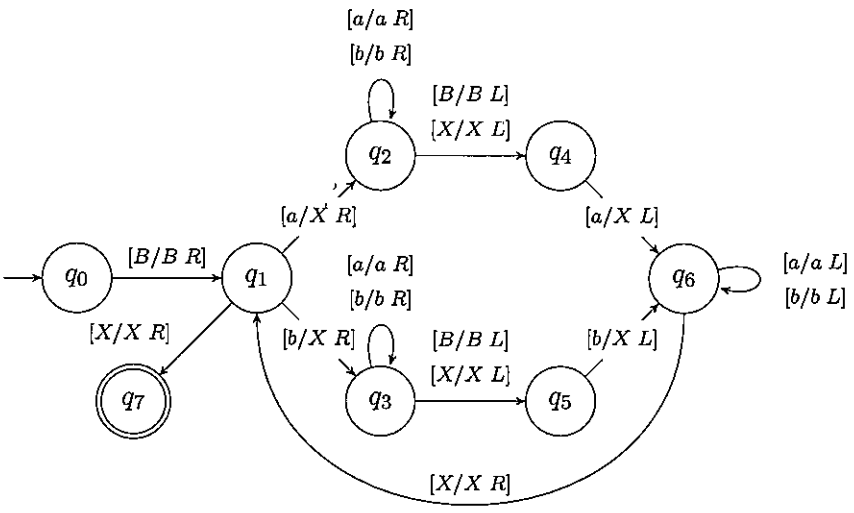
$$G_2 = \begin{cases} S & \to & BS \mid AB \\ A & \to & Aaa \mid \lambda \\ B & \to & b \end{cases}$$

3. (5 points) Consider the context-free language $L = \{a^i b^{i+j} c^j \mid i, j > 0\}$.
Give a DPDA (*deterministic* pushdown automaton) for L. Provide a *short* explanation.

4. (5 points) Let G be a context-free grammar in Greibach Normal Form. Let P be a deterministic pushdown automaton, and let E be a regular expression.
Is the language $\mathcal{L}(G) \cup (\mathcal{L}(P) \cap \mathcal{L}(E))$ context-free? (prove)

5. (5 points) Is the class of recursive languages closed under complement? (prove)

6. (5 points) Consider the following Turing Machine with a single tape.
Which language is *decided* by this TM? (explain shortly)



Algebra

5. Are $U(14)$ and $U(18)$ isomorphic?

6. (a) Let $\sigma \in S_8$ be given by

$$\sigma = (1586)(23)$$

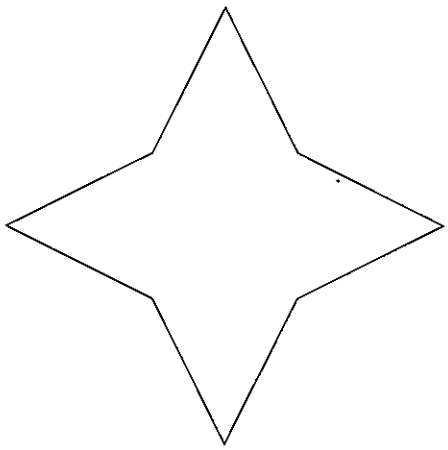
Determine whether σ is even or odd.

(b) Let $\tau \in S_5$ be given by

$$\tau = (12)(134)(152)$$

What is the order of τ?

7. Use Burnside's theorem to determine the number of different ways in which the edges of a star (see figure), made of copper wire, can be colored using two colors.



8. (a) Show that $a(x) = x^2 + x + 1$ is the only irreducible polynomial of degree two in $\mathbb{Z}_2[x]$.
   (b) Expand $(a(x))^2$ in $\mathbb{Z}_2[x]$.
   (c) Show that $p(x) = x^4 + x^3 + 1 \in \mathbb{Z}_2[x]$ is irreducible.
   (d) Define
   $$\mathbb{F} = \mathbb{Z}_2[x]/ < x^4 + x^3 + 1 > .$$
   Argue that $\mathbb{F}$ is a field.
   (e) How many elements does $\mathbb{F}$ have?
   (f) Determine the multiplicative order of $x+ < x^4 + x^3 + 1 > \in \mathbb{F}$.

Points: **Ex 5**: 10; **Ex 6**: a: 4, b: 6; **Ex 7**: 12, **Ex 8**: a: 3, b: 2, c: 4, d: 2, e: 3, f: 4.

Discrete Mathematics

9. (7 points) Consider the RSA method, and assume that Alice has published the modulus $n = 91$ and the exponent $e = 35$. Bob emails the cipher text $C = 3$ to Alice. Compute everything that Alice needs to compute Bob's original message M, and also compute M.

10. (3 points) In the year 2002 the indian researcher Agrawal together with his BSc students Kayal and Saxena published a breakthrough paper in which they solve a long-standing open problem, namlely a polynomial time algorithm that correctly solves the following decision problem: Given a number $n \in \mathbb{N}$, is n a prime or not? Does that put the RSA crypto system in danger? (Please give a *short* but conclusive answer; "yes" or "no" does not suffice.)