Discrete Mathematics for Computer Science, January 14, 2021; Part 1
Solution/Correction standard

1. (a) $$\forall x \in U\colon\ x \notin A \vee x \notin B$$ **[3 pt]**

$$\forall x \in A\colon x \notin B \wedge \forall x \in B\colon x \notin A.$$ **[3 pt]**

(b) $$\forall x \in C\colon x \in a \wedge \exists x \in B\colon x \notin B.$$ **[3 pt]**

For each expression that is not logically equivalent to the ones above: 0 pt.

2.

(1)	q	Premise
(2)	$q \to (u \wedge s)$	Premise
(3)	$u \wedge s$	(1), (2), R1
(4)	u	(3), R7
(5)	$u \to r$	Premise
(6)	r	(4), (5), R1
(7)	s	(3), L3, R1
(8)	$r \wedge s$	(6), (7), R4
(9)	$(r \wedge s) \to (p \vee t)$	Premise
(10)	$p \vee t$	(8), (9), R1
(11)	$\neg t$	Premise
(12)	p	(10), (11), L3, R5

[6 pt]

For each forgotten Law or Rule: -0.5 pt.
If deduction contains a step that is not logically correct: at most 1 pt for the entire exercise.

3. We show that $(A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B)$ and $(A \cup B) - (A \cap B) \subseteq (A-B) \cup (B-A)$. **[1 pt]**

(i) Proof of $(A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B)$.
Let $x \in (A-B) \cup (B-A)$. Then $x \in (A-B) \vee x \in (B-A)$ **[0.5 pt]**
$x \in (A-B) \to x \in A \wedge x \notin B \to x \in A \cup B \wedge x \notin B \cap A$ **[1 pt]**
$x \in (B-A) \to x \in B \wedge x \notin A \to x \in B \cup A \wedge x \notin A \cap B$. **[1 pt]**
Hence, in both cases $x \in (A \cup B) - (A \cap B)$. **[0.5 pt]**

(ii) Proof of $(A \cup B) - (A \cap B) \subseteq (A-B) \cup (B-A)$.
Let $x \in (A \cup B) - (A \cap B)$. Then $(x \in A \vee x \in B) \wedge x \notin A \cap B$. **[0.5 pt]**
So $x \in A \wedge x \notin A \cap B$ or $x \in B \wedge x \notin A \cap B$. **[0.5 pt]**
$x \in A \wedge x \notin A \cap B \to x \in A \wedge x \notin B \to x \in A - B$.
$x \in B \wedge x \notin A \cap B \to x \in B \wedge x \notin A \to x \in B - A$. **[0.5 pt]**
So $x \in (A-B) \cup (B-A)$. **[0.5 pt]**

Discrete Mathematics for Computer Science, January 14 2021; Part 2
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4. Basis step for $n = 2, 3$:

$$a_2 = \frac{1}{2} \cdot a_1 + \frac{1}{2} \cdot a_0 = \frac{1}{2} \text{ and } 0 \leq \frac{1}{2} \leq 1$$ **[0.5 pt]**

$$a_3 = \frac{1}{3} \cdot a_2 + \frac{2}{3} \cdot a_1 = \frac{5}{6} \text{ and } 0 \leq \frac{5}{6} \leq 1$$ **[1 pt]**

Induction step: Let $k \geq 3$ and suppose that for all $p \in \{2, 3, ..., k\}$: **[1 pt]**

$$0 \leq a_p \leq 1$$ **[0.5 pt]**

We must show that the induction hypothesis implies: **[0.5 pt]**

$$0 \leq a_{k+1} \leq 1.$$ **[0.5 pt]**

By the definition of the recursion we can write:

$$a_{k+1} = \frac{1}{k+1} \cdot a_k + \frac{k}{k+1} \cdot a_{k-1}$$ **[0.5 pt]**

By the IH, we have that: **[0.5 pt]**

$$0 \leq \frac{1}{k+1} \cdot a_k \leq \frac{1}{k+1} \text{ and } 0 \leq \frac{k}{k+1} \cdot a_{k-1} \leq \frac{k}{k+1}$$ **[0.5 pt]**

And hence, we can write:

$$0 \leq a_{k+1} \leq \frac{1}{k+1} + \frac{k}{k+1} = 1$$ **[0.5 pt]**

(From the proof it must be crystal clear whàt is supposed **[1 pt]** and whàt must be proved **[1 pt]**. In case the induction hypothesis is not correctly formulated or the proof is not clearly written down: at most **1 pt** for the entire exercise)

5. We will prove that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ and that $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$. **[1 pt]**

Proof of $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$:
Let $b \in f(A_1 \cap A_2)$. Then we know that there is an $x \in A_1 \cap A_2$ such that $f(x) = b$. **[0.5 pt]**
Since $x \in A_1 \cap A_2$, we have that $x \in A_1$ and $x \in A_2$. **[0.5 pt]**
So $b \in f(A_1)$ and $b \in f(A_2)$ and hence $b \in f(A_1) \cap f(A_2)$. **[1 pt]**

Proof of $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$:
Let $b \in f(A_1) \cap f(A_2)$. Then we know that there exist $x \in A_1$ and $y \in A_2$ such that $f(x) = f(y) = b$. **[1 pt]**
Since f is one-to one, we must have that $x = y$ because x and y have the same image in B under f. **[1 pt]**
So $x \in A_1 \cap A_2$ and hence $b \in f(A_1 \cap A_2)$. **[1 pt]**

6. (a) (i) R is reflexive since $|x-x| = 0 \in \mathbb{Z}$ **[1 pt]**

(ii) R is symmetric because if $|x-y| \in \mathbb{Z}$, then we have that $|y-x| \in \mathbb{Z}$ because $|x-y| = |y-x|$ (since $y-x = -(x-y)$). **[1 pt]**

(iii) R is transitive because if $|x-y| \in \mathbb{Z}$ and $|y-z| \in \mathbb{Z}$, we have that $|x-z| \in \mathbb{Z}$

$|x-y| \in \mathbb{Z} \to x-y \in \mathbb{Z}$ and $|y-z| \in \mathbb{Z} \to y-z \in \mathbb{Z}$. **[1 pt]**

So $x-z \in \mathbb{Z}$ since $x-z = x-y+y-z$. **[0.5 pt]**

So $x-z \in \mathbb{Z} \to |x-z| \in \mathbb{Z}$. **[0.5 pt]**

(b) If $|x-y| \in \mathbb{Z}$, then $x-y \in \mathbb{Z}$
That means that $x = y+k$ where k is an integer. **[1 pt]**
So we have that $[1.5] = \{1.5+k,\ k \in \mathbb{Z}\}$ **[1 pt]**