Reference : AM2017/DWMP/004/ha

Course : **Discrete Mathematics for Computer Science**
Date : November 10, 2017
Time : 08.45–10.45 hrs

Motivate all your answers. A formula sheet is included.
The use of electronic devices is not allowed.

In this exam: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

1. Let A be an $m \times n$-matrix with $a_{ij} \in \{0, 1\}$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Give quantified expressions for the following statements.

   (a) [3 pt] In each row the zeros and ones alternate (like $010101\cdots$ or $101010\cdots$).

   (b) [3 pt] Each column contains at least two zeros.

2. [6 pt]
   Prove the following equivalence using the Laws of Logic.

$$\neg (p \leftrightarrow q) \iff p \leftrightarrow \neg q.$$

3. [6 pt]
   Let A, B and C be sets in a universe $\mathcal{U}$.

   (a) [4 pt] Prove that $(A \cup B) - C \subseteq (A - B) \cup (B - C)$.

   (b) [2 pt] Show with a counterexample that the converse inclusion $(A - B) \cup (B - C) \subseteq (A \cup B) - C$ is not necessarily true.

4. [6 pt]
   Prove with mathematical induction that for all $n \in \mathbb{N}$,

$$2^{3n+4} + 3^{3n+1} \quad \text{is divisible by } 19.$$

5. [6 pt]
   Let A, B and C be sets and let $f : A \to B$ and $g : B \to C$ be functions such that g is one-to-one and $g \circ f$ is onto. Prove that f is onto.

6. Let (A, R) be a poset and let $B \subseteq A$.

   (a) [4 pt] Prove that B has at most one greatest lower bound.

   (b) [2 pt] Give an example of a poset (A, R) and a set $B \subseteq A$ such that B has at least one lower bound but has no greatest lower bound.

Total: 36 points

Laws of Logic

L1.	$\neg\neg p \Leftrightarrow p$	Law of Double Negation
L2.	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	DeMorgan's Laws
L3.	$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	Commutative Laws
L4.	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	Associative Laws
L5.	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive Laws
L6.	$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$	Idempotent Laws
L7.	$p \vee F_0 \Leftrightarrow p$ $p \wedge T_0 \Leftrightarrow p$	Identity Laws
L8.	$p \vee \neg p \Leftrightarrow T_0$ $p \wedge \neg p \Leftrightarrow F_0$	Inverse Laws
L9.	$p \vee T_0 \Leftrightarrow T_0$ $p \wedge F_0 \Leftrightarrow F_0$	Domination Laws
L10.	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	Absorption Laws
L11.	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$	
L12.	$p \rightarrow q \Leftrightarrow \neg p \vee q$	
L13.	$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$	

Rules of Inference

R1.	p $p \rightarrow q$ $\therefore q$	Modus Ponens
R2.	$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	Law of the Syllogism
R3.	$p \rightarrow q$ $\neg q$ $\therefore \neg p$	Modus Tollens
R4.	p q $\therefore p \wedge q$	Rule of Conjunction
R5.	$p \vee q$ $\neg p$ $\therefore q$	Rule of Disjunctive Syllogism
R6.	$\neg p \rightarrow F_0$ $\therefore p$	Rule of Contradiction
R7.	$p \wedge q$ $\therefore p$	Rule of Conjunctive Simplification
R8.	p $\therefore p \vee q$	Rule of Disjunctive Amplification
R9.	$p \wedge q$ $p \rightarrow (q \rightarrow r)$ $\therefore r$	Rule of Conditional Proof
R10.	$p \rightarrow r$ $q \rightarrow r$ $\therefore (p \vee q) \rightarrow r$	Rule for Proof by Cases
R11.	$p \rightarrow q$ $r \rightarrow s$ $p \vee r$ $\therefore (q \vee s)$	Rule of the Constructive Dilemma
R12.	$p \rightarrow q$ $r \rightarrow s$ $\neg q \vee \neg s$ $\therefore \neg p \vee \neg r$	Rule of the Destructive Dilemma

Laws concerning quantifiers

N1. $\neg[\forall x\, p(x)] \Leftrightarrow \exists x\, \neg p(x)$

N2. $\neg[\exists x\, p(x)] \Leftrightarrow \forall x\, \neg p(x)$

Additional Laws concerning quantifiers

U1.
$$\frac{\forall x\, p(x)}{\therefore p(c) \text{ for arbitrary } c \text{ in the universe}}$$

U2.
$$\frac{\exists x\, p(x)}{\therefore p(c) \text{ for some } c \text{ in the universe}}$$

U3.
$$\frac{p(c) \text{ for arbitrary } c \text{ in the universe}}{\therefore \forall x\, p(x)}$$

U4.
$$\frac{p(c) \text{ for some } c \text{ in the universe}}{\therefore \exists x\, p(x)}$$

U1: Rule of Universal Specification

U2: Rule of Existential Specification

U3: Rule of Universal Generalization

U4: Rule of Existential Generalization