Test Probability Theory, 26 July 2018.

1. W: the firm wins the bid
E: the firm is asked for extra information
$P(W) = 0.60$, $P(E|W) = 0.75$, $P(E|\bar{W}) = 0.40$.
Asked: $P(W|E)$.
Bayes' rule applies:

$$P(W|E) = \frac{P(WE)}{P(E)} = \frac{P(E|W)\cdot P(W)}{P(E|W)\cdot P(W) + P(E|\bar{W})\cdot P(\bar{W})}$$

$$= \frac{0.75\cdot 0.60}{0.75\cdot 0.60 + 0.40\cdot 0.40} = 0.738.$$

2. a) $X \sim B(10, 0.5)$, so using the table:
$P(X \geq 8) = 1 - P(X \leq 7) = 1 - 0.945 = 0.055$
b) $E(X) = np = 10\cdot 0.5 = 5$.
c) $var(X) = np(1-p) = 10\cdot 0.5\cdot 0.5 = 2.5$.

3. a) The marginal distributions of X and Y are added to the table:

$x \backslash y$	0	1	4	$P(X=x)$ (row-sums)
−1	0.04	0.10	0.15	0.29
0	0.16	0.05	0.10	0.31
1	0.20	0.10	0.10	0.40
$P(Y=y)$ (column-sums)	0.40	0.25	0.35	

b) $cov(X,Y) = E(XY) - E(X)E(Y)$
$E(X) = -1\cdot 0.29 + 0\cdot 0.31 + 1\cdot 0.40 = 0.11$
$E(Y) = 0\cdot 0.40 + 1\cdot 0.25 + 4\cdot 0.35 = 1.65$
$E(XY) = (-1)\cdot 1\cdot 0.10 + (-1)\cdot 4\cdot 0.15 + 1\cdot 1\cdot 0.10 + 1\cdot 4\cdot 0.10$
$= -0.20$, so $cov(X,Y) = -0.20 - (0.11)(1.65) \approx -0.38$

c) $P(X^2 + Y = 1) = P(X=-1 \text{ and } Y=0) + P(X=0 \text{ and } Y=1)$
$+ P(X=1 \text{ and } Y=0)$
$= 0.04 + 0.05 + 0.20 = 0.29.$

3d) $P(Y=y \mid X=1) = P(Y=y \text{ and } X=1) / P(X=1)$.

$E(X \mid Y=1)$

$= 0 \cdot P(Y=0 \mid X=1) + 1 \cdot P(Y=1 \mid X=1) + 4 \cdot P(Y=4 \mid X=1)$

$= 0 + 1 \cdot 0.10/0.40 + 4 \cdot 0.10/0.40 = 1.25$.

4. (a) $E(X) = \frac{a+b}{2} = 1 \iff a = 2-b$

formula sheet $\quad \text{var}(X) = \frac{(b-a)^2}{12} = \frac{4}{3} \iff (2b-2)^2 = 16 \iff 2b-2 = \pm 4$

$\iff b = 1 \pm 2$.

If $b = -1$ then $a = 3 > b$: not possible!

So, $b = 3$ and $a = -1$.

b) Density $f_X(x) = \frac{1}{4}$ if $-1 \le x \le 3$, and 0 elsewhere.

c: 10^{th} percentile of X. Then

$$0.10 = P(X \le c) = \int_{-1}^{c} \frac{1}{4}\, dx = \left[\tfrac{1}{4}x\right]_{-1}^{c} = \tfrac{1}{4}(c+1)$$

$\iff c = -0.60$.

c) $F_Y(y) = P(Y \le y) = P(X^3 \le y) = P(X \le y^{\frac{1}{3}}) = F_X(y^{\frac{1}{3}})$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(y^{\frac{1}{3}}) = f_X(y^{\frac{1}{3}}) \cdot \frac{1}{3} y^{-\frac{2}{3}}$$

$$= \begin{cases} \frac{1}{4} \cdot \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{12} y^{-\frac{2}{3}}, & -1 \le y^{\frac{1}{3}} \le 3 \iff -1 \le y \le 27 \\ 0, & \text{otherwise.} \end{cases}$$

5. a) X describes random draws without replacement from a dichotomous population. Therefore, the hypergeometric distribution applies.

b)

	HP	mac	Total
population	12	8	20
	↓	↓	↓
sample	2	4	6

$$P(X=2) = \frac{\binom{12}{2}\binom{8}{4}}{\binom{20}{6}} = \frac{77}{646} \approx 0.1192$$

c) For relatively small samples out of large populations, the hypergeometric distribution is approximately equal to the binomial distribution.

Rule of thumb: $4,800 = N \geq 5n^2 = 5 \cdot 30^2 = 4,500$.

d) By (c), $Y \sim B(30, \frac{2880}{4800}) = B(30, 0.6)$ approximately.

By the CLT $Y \sim N(30 \cdot 0.6, 30 \cdot 0.6 \cdot 0.4) = N(18, 7.2)$ approximately.

$P(Y \leq 12) \overset{c.c.}{=} P(Y \leq 12.5) \overset{CLT}{\approx} P\left(Z \leq \frac{12.5-18}{\sqrt{7.2}}\right)$

$= \Phi(-2.05) = 1 - \Phi(2.05) = 0.0202$.

6. a) $E(X+Y) = E(X) + E(Y) = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} = 1.5$.

$var(X+Y) = var(X) + var(Y) + 2\, cov(X,Y)$ (formula sheet)

$= 1/1^2 + 1/2^2 + 0$ (independence) $= 5/4 = 1.25$.

b) $\rho(X, X+Y) = \frac{cov(X, X+Y)}{\sigma_X \cdot \sigma_{X+Y}} = \frac{cov(X,X) + cov(X,Y)}{\sqrt{var(X)} \cdot \sqrt{var(X+Y)}}$

$= \frac{var(X) + 0}{\sqrt{1} \cdot \sqrt{1.25}} \approx 0.8944$.

c) According to the CLT $(n > 25)$ $X_1 + \ldots + X_{100}$ is approximately $N(n \cdot \frac{1}{\lambda}, n \cdot \frac{1}{\lambda^2}) = N(100, 100)$ - distributed.

Similarly, $Y_1 + \ldots + Y_{100} \sim N(50, 25)$.

Because all variables are independent, the sum (of the two sums) is also normally distributed.

The total sum of service times $(X_1 + \ldots + X_{100}) + (Y_1 + \ldots + Y_{100})$ is thus approximately $N(100+50, 100+25) = N(150, 125)$ distributed.