201700080 Information Theory and Statistics
26 June 2023, 13:45 – 16:45

This test consists of 5 problems for a total of 28 points. All answers need to be justified. The use of a non-programmable calculator (not a "GR") is allowed. A handwritten single side A4 cheat sheet is allowed. No additional books or notes may be used.

1. You receive a sample of n observations $x_1, x_2, \ldots, x_n$, that are independent and identically distributed according to a Bernoulli distribution for which the success probability is unknown. It is known that either $p = p_1$ or $p = p_2$, i.e. the distribution is either $P_1(X = x) = (1 - p_1)^{1-x} p_1^x$ or $P_2(X = x) = (1 - p_2)^{1-x} p_2^x$, $x \in \{0, 1\}$.

   a. [3 pt] Specify a binary hypothesis testing problem choosing between p_1 and p_2. Derive an optimal decision rule.

   b. [1 pt] Compute $D(P_1 || P_2)$.

   c. [2 pt] State the Chernoff-Stein result for this problem and discuss its meaning.

2. Let X and Y be random variables that take values $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ and $\mathcal{Y} = \{y_1, y_2, \ldots, y_m\}$, respectively. Let $Z = X + Y$.

   a. [2 pt] Prove that $H(Z|X) = H(Y|X)$.

   b. [2 pt] Prove that, if X and Y are independent, $H(X|Y) = X$. $H(X)$

   c. [1 pt] Use a) and b) to prove that, if X and Y are independent, $H(Z) \geq H(X)$ and $H(Z) \geq H(Y)$.

3. Let X be an exponentially distributed random variable with rate parameter $\theta > 0$, i.e. $f_\theta(x) = \theta e^{-\theta x}, x \geq 0$. Consider the estimator $\hat{\theta}_n = \frac{n}{\sum_{j=1}^n x_j}$, for estimating θ based on observations $x_1, x_2, \ldots, x_n$. It is known (and you can use these facts in your solutions) that this estimator is unbiased and efficient.

   a. [2 pt] Compute $\mathbb{E}[\hat{\theta}_n]$.

   b. [3 pt] Compute the Fisher information $J(\theta)$.

   c. [3 pt] Compute $\text{var}(\hat{\theta}_n)$. State theoretical results that you are using and explain.

4. [3 pt] A dictionary-based Lempel-Ziv code uses λ to denote the empty string and σ to denote the end-of-message symbol. It gives the following compressed message:

$$(\lambda, a), (1, a), (\lambda, b), (3, c), (3, b), (2, a), (\lambda, \sigma).$$

   Decode and give the original message. Explain your steps.

5. Consider the channel capacity with $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and $P(Y = 0|X = 0) = 1$ and $P(Y = 0|X = 1) = \frac{1}{2}$. In this exercise, use the natural logarithm in all expressions.

   a. [3 pt] Let $P(X = 1) = p$. Give expressions for $H(Y)$ and $H(Y|X)$.

   b. [3 pt] Compute the capacity of the channel. Give a numerical answer with two decimals precision. *(Hint: the derivative of* $-\frac{1}{2}x\log(\frac{1}{2}x) - (1 - \frac{1}{2}x)\log(1 - \frac{1}{2}x)$ *is equal to* $\frac{1}{2}\log(\frac{2-x}{x})$*)*