Kenmerk: EWI2019/TW/DMMP/MU/Mod7/Re-Exam1

Re-Exam 1, Module 7, Codes 201400483 & 201800141

Discrete Structures & Efficient Algorithms

Tuesday, April 16, 08:45 - 11:45

All answers need to be motivated. No calculators. You are allowed to use a handwritten cheat sheet (A4, both sides) per topic (ADS, DM).

This exam consists of two parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	ca. 1h	(30 points)
Discrete Mathematics (DM)	ca. 2h	(60 points)

The total is 30+60=90 points. Your exam grade is 1 plus and the total number of points multiplied by 0.1 (= 9/90), rounded to one digit.

Important: It is necessary to use a new sheet of paper for each part (ADS and DM)!

Algorithms & Data Structures

1. (10 points)

   Consider this sorting algorithm that sorts from a sequence A of integers the segment $A[i,\ldots,j]$ where $1 \leq i \leq j$:

```
def sort(A,i,j):
    if A[i]>A[j] : A[i],A[j]=A[j],A[i]
    if i+1>=j : return
    k=(j-i+1)//3
    sort(A,i,j-k)
    sort(A,i+k,j)
    sort(A,i,j-k)
```

   (a) Determine the asymptotic worst-case complexity for *sort* to sort $n > 0$ numbers. Take as basic operation the comparison of elements of A.

   (b) Under which circumstances would you prefer *sort* over quicksort, insertion sort, mergesort or heapsort?

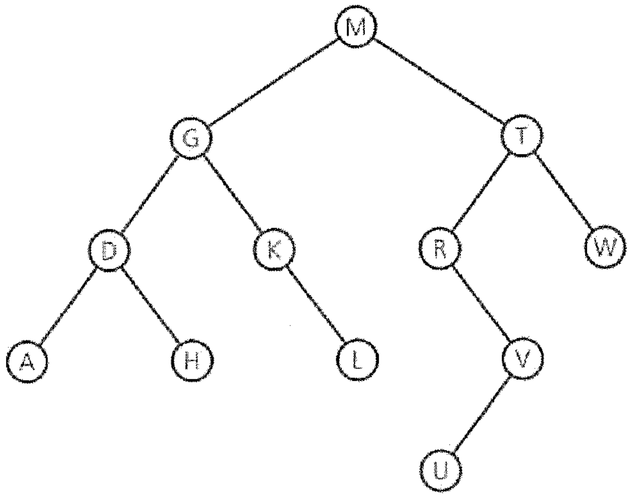
2. (5 points)

   (a) Where is the smallest element in a maxheap?

   (b) Given an array A sorted in decreasing order. Give an efficient algorithm that turns A into a heap.

3. (5 points)

Given the following binary tree:



Write down (as a string of letters) the order of the nodes you encounter for the preorder, inorder, and postorder traversal of the tree.

4. (10 points) Given a $M \times N$ matrix where each cell has a cost associated to it. We are interested in the minimum cost to reach cell $(M-1, N-1)$ starting from cell $(0,0)$ where you can only move one unit right or one down from any cell, i.e. from cell (i,j) you can move to $(i, j+1)$ or $(i+1, j)$.

   (a) We want to fill a matrix T such that $T(i,j)$ contains the minimum cost for going from $(0,0)$ to (i,j). Explain that for $j > 0$, $T(0,j) = cost(0,j) + T(0,j-1)$, and for $i > 0$, $T(i,0) = cost(i,0) + T(i-1,0)$. Give an expression for $T(i,j)$ for $i,j > 0$.

   (b) Give an algorithm that gives the minimum cost for going from $(0,0)$ to $(M-1, N-1)$. The algorithm should have complexity not bigger than $\Theta(MN)$.

Discrete Mathematics

5. (10 points)

   (a) By using the Euclidean algorithm, show that $708s + 72t = 4$ has no solution for $s, t \in \mathbb{Z}$.

   (b) Let a and b be coprime, and $a > b \geq 0$. Define $d := \gcd(a-b, a+b)$. Show that $d \leq 2$.

6. (10 points)

   (a) Let us denote by a_n the number of strings in $\{0,1,2\}^*$ of length n where there are no consecutive 1's and no consecutive 2's. Compute a_1 and a_2, and set up a recurrence relation for a_n $(n \geq 3)$. You do not need to solve this recurrence relation.

(b) Compute the solution to the recurrence relation

$$a_n - 6a_{n-1} + 9a_{n-2} = 4n + 4 \quad (n \geq 2) \quad \text{with } a_0 = 5 \text{ and } a_1 = 9\,.$$

7. (10 points)

   (a) Suppose we want to donate 100€ to three charity organizations C_1, C_2, C_3, such that each of them gets at least 20€, but at most 50€, and moreover each organization gets an integer amount. How many different possibilities are there to do that? Use a generating function to compute your answer.

   (b) If the question is to count the number of different possibilities of splitting up 100€ into three parts, such that each part is an integer amount, at least 20€ and at most 50€, is the answer

   - smaller than
   - larger than
   - equal to

   the answer in (a)?

8. (10 points) Let $G = (V, E)$ be a simple, connected, undirected graph with $|V| = n$ and $|E| = m$ without a bridge. Show that, if at least half of the nodes of G have a degree at least 10, then G cannot be planar.

9. (14 points) Suppose we are given a capacitated network $G = (V, E, c)$, where V is the set of vertices, E is the set of (directed) edges, and $c(e) \geq 0$, $e \in E$ are the arc capacities. Also, let $s, t \in V$ and $f : E \to \mathbb{R}$ be a feasible flow in G. Give a short proof or give a counterexample:

   (a) Multiplying each of the capacities $c(e)$ by a number $\lambda > 0$ does not change set of minimal (s,t)-cuts.

   (b) Adding a number $\mu > 0$ to each of the capacities c_a does not change the set of minimal (s,t)-cuts.

   Now consider an undirected graph $G = (V, E)$ with integer arc weights $w(e) \geq 0$, $e \in E$. Give a short proof or give a counterexample:

   (c) If T_1 and T_2 are two minimum spanning trees for G, then $\max\{w_e \mid e \in T_1\} = \max\{w_e \mid e \in T_2\}$.

   (d) If T_1 and T_2 are two minimum spanning trees for G, then $T_1 \cap T_2 \neq \emptyset$.

10. (6 points) Consider a simple, capacitated network $G = (V, E, c)$, where V is the set of nodes, $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$ for $e \in E$ are the edge capacities. Let $n = |V|$ and $m = |E|$. Suppose you are given a maximum (s,t)-flow f for G. Suggest how to compute a minimum (s,t)-cut (S,T) for G in computation time $O(n+m)$.