Solution/Correction standard, 1st Test Mathematics B1; October 24, 2014.

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Solution/Correction standard

1. [3 pt] Method 1

The differential equation is linear, it is of the form $y' + P(x)y = Q(x)$ with $P(x) = 3x^2$ and $Q(x) = 0$. The integrating factor is

$$v(x) = e^{\int P(x)\, \mathrm{d}x} = e^{\int 3x^2\, \mathrm{d}x} = e^{x^3}. \qquad \textbf{[1 pt]}$$

Multiply the equation with $v(x)$: The general solution is

$$\begin{aligned} v(x)y' + v(x) \cdot 3x^2 y &= 0 \\ v(x)y' + v'(x)y &= 0 \\ \frac{d}{dx}\big(v(x)y(x)\big) &= 0 \\ v(x)y(x) &= C, \end{aligned}$$

hence $y(x) = \dfrac{C}{v(x)} = Ce^{-x^3}$. **[1 pt]**

Remark: For the derivation **[0.5 pt]** and for the answer **[0.5 pt]**.

Use the initial condition to find C:

$$5 = y(0) = Ce^0 = C \quad \Rightarrow \quad C = 5.$$

The solution therefore is:

$$y(x) = 5e^{-x^3}. \qquad \textbf{[1 pt]}$$

Alternative Method, not treated in the lectures.

Basically, the equation is treated as a separable differential equation. The equation can be rewritten as

$$\frac{y'}{y} = -3x^2.$$

Integrate left and right hand side, or notice that y'/y is the logarithmic derivative of y:

$$\ln|y(x)| = -x^3 + K$$

with real constant K. From this follows

$$|y(x)| = e^{-x^3+K} = e^{-x^3}e^K = C_+e^{-x^3} \qquad \textbf{[1 pt]}$$

with real positive constant $C_+ = e^K$. Since solutions must be continuous, taking the absolute value means

$$y(x) = \pm C_+ e^{-x^3} = Ce^{-x^3} \tag{1}$$

with non-zero constant C.

Since the constant function $y(x) = 0$ is also a solution, (1) holds for *every* real constant C. **[1 pt]**

See method 1 for finding the value of C. **[1 pt]**

2. (a) [2 pt] For the modulus and the argument find real and imaginary part of z:

$$z = \frac{\sqrt{2}\,i}{1-i} = \frac{\sqrt{2}\,i}{1-i}\,\frac{1+i}{1+i} = \frac{1}{2}\sqrt{2}(i-1).$$ **[1 pt]**

Hence the modulus equals

$$|z| = \sqrt{\left(\frac{-1}{2}\sqrt{2}\right)^2 + \left(\frac{1}{2}\sqrt{2}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1.$$ **[0.5 pt]**

Alternatively, you may use that $|z|^2 = z\overline{z}$:

$$z\overline{z} = \frac{\sqrt{2}\,i}{1-i}\,\frac{\sqrt{2}(-i)}{1+i} = \frac{-2i^2}{1^2+1^2} = 1,$$

hence $|z| = \sqrt{1} = 1$. **[0.5 pt]**

For the argument, there are also two alternatives.

Alternative 1:

For the argument θ we have

$$\tan\theta = \frac{\operatorname{Im} z}{\operatorname{Re} z} = \frac{-\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}} = -1.$$

This means that $\theta = -\dfrac{\pi}{4}$ or $\theta = \dfrac{3\pi}{4}$. From $\operatorname{Re} z < 0$ we readily conclude that $\theta = \dfrac{3\pi}{4}$. **[0.5 pt]**

Alternative 2:

Use a picture, see figure 1. **[0.5 pt]**

(b) [1 pt] Since i and z lie in the unit circle, the question is equivalent to the following problem: does there exist a positive integer n for which $n\cdot\frac{3}{4}\pi = \frac{1}{2}\pi + 2k\pi$ for some integer k? **[0.5 pt]**

Solving for n gives the equation

$$n = \frac{8k+2}{3}.$$

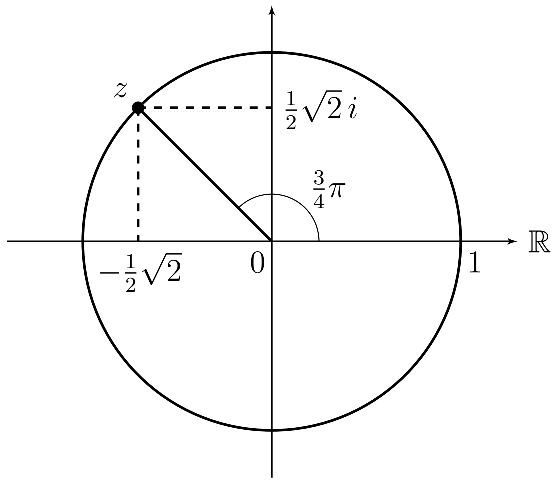

Figure 1: z lies on the unit circle (assignment 2).

We are therefore looking for a value of k for which $8k+2$ is a multiple of 3. Make a table to find the appropriate value of k.

k	$8k+2$	multiple of 3	n
0	2	no	–
1	10	no	–
2	18	yes	6

So the answer is "yes", and the smallest positive integer n for which $z^n = i$ is $n = 6$. **[0.5 pt]**

3. [1 pt] Let $z = x + iy$ with $x, y \in \mathbb{R}$. Then $|z| = \sqrt{x^2+y^2}$. **[0.5 pt]**

For $|\overline{z}|$ we then have

$$\begin{aligned} |\overline{z}| &= |x - iy| \\ &= \sqrt{x^2 + (-y)^2} \\ &= \sqrt{x^2+y^2} = |z|\,. \end{aligned}$$ **[0.5 pt]**

Alternatively, one can write

$$\begin{aligned} |\overline{z}|^2 &= |x - iy|^2 \\ &= x^2 + (-y)^2 \\ &= x^2 + y^2 = |z|^2\,. \end{aligned}$$

Taking square roots then gives $|z| = |\overline{z}|$. **[0.5 pt]**

Alternative 2: Any complex number z_0 can be uniquely written as $z_0 = r_0 e^{i\varphi_0}$ with $r_0 = |z_0|$ and $\varphi_0 = \arg(z_0)$. **[0.5 pt]**

Hence $z = re^{i\varphi}$ and so $\overline{z} = re^{-i\varphi}$. Thus $|z| = r = |\overline{z}|$. **[0.5 pt]**

4. [5 pt] Step 1 (total: 2 pt): solve the homogeneous equation $y'' + 2y' + 10y = 0$.

The corresponding auxiliary or characteristic equation is $\lambda^2 + 2\lambda + 10 = 0$. This equation has two imaginary roots:

$$\lambda = -1 + 3i \quad \text{and} \quad \lambda = -1 - 3i. \qquad \textbf{[1 pt]}$$

Therefore the general real solution of the homogeneous equation is

$$y(x) = e^{-x}\big(c_1 \cos(3x) + c_2 \sin(3x)\big). \qquad \textbf{[1 pt]}$$

Alternatively, the general complex solution of the homogeneous equation is

$$y(x) = c_1 e^{(-1+3i)x} + c_2 e^{(-1-3i)x} \qquad \textbf{[0.5 pt]}$$

Remark: The remaining half point is for writing the final answer in real form.

Step 2 (total: 1 pt): find a particular solution.

We try a multiple of e^x, in other words: define

$$y_p(x) = a\, e^x \qquad \textbf{[0.5 pt]}$$

with unknown constant a. Notice that

$$y_p''(x) = y_p'(x) = a\, e^x,$$

hence

$$y_p'' + 2y_p' + 10y_p = 13a\, e^x$$

From this readily follows that $a = 1$. **[0.5 pt]**

Step 3 (total: 2 pt): determine the constants c_1 and c_2.

The general solution is

$$y(x) = e^{-x}(c_1 \cos(3x) + c_2 \sin(3x)) + e^x. \qquad (1)$$

From $y(0) = 1$ follows

$$1 = y(0) = 1 \cdot (c_1 \cdot 1 + c_2 \cdot 0) + 1 = c_1 + 1,$$

hence $c_1 = 0$. **[1 pt]**

The general solution (1) simplifies to

$$y(x) = c_2\, e^{-x} \sin(3x) + e^x.$$

Differentiate $y(x)$:

$$y'(x) = c_2\big(-e^{-x} \sin(3x) + 3e^{-x} \cos(3x)\big) + e^x.$$

From $y'(0) = 4$ follows

$$4 = y'(0) = c_2(-1 \cdot 0 + 3 \cdot 1 \cdot 1) + 1 = 3c_2 + 1,$$

hence $c_2 = 1$. **[1 pt]**

The solution of the initial value problem is

$$y(x) = e^{-x} \sin(3x) + e^x.$$

5. (a) [2 pt] Calculate the cross product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{q} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \end{bmatrix} \begin{matrix} 1 & -1 \\ -1 & 2 \end{matrix} = \langle 5, 4, 1 \rangle.$$ **[2 pt]**

(b) [1 pt] <u>Method 1</u>: Notice that by one of the defining properties of the cross product there holds $\mathbf{u} \perp \mathbf{q}$ and $\mathbf{u} \perp \mathbf{u} \times \mathbf{w}$. So any non-zero multiple of $\mathbf{u}$ will do the trick. **[1 pt]**

<u>Method 2</u>: First calculate $\mathbf{u} \times \mathbf{w}$:

$$\mathbf{u} \times \mathbf{w} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & -5 \end{bmatrix} \begin{matrix} 1 & -1 \\ 1 & 3 \end{matrix} = \langle 8, 4, 4 \rangle.$$ **[0.5 pt]**

Now by construction $\mathbf{q} \times (\mathbf{u} \times \mathbf{w})$ will be orthogonal to $\mathbf{q}$ and $\mathbf{u} \times \mathbf{w}$.

$$\mathbf{q} \times (\mathbf{u} \times \mathbf{w}) = \begin{bmatrix} 5 & 4 & 1 \\ 8 & 4 & 4 \end{bmatrix} \begin{matrix} 5 & 4 \\ 8 & 4 \end{matrix} = \langle 12, -12, -12 \rangle = 12\mathbf{u}.$$ **[0.5 pt]**

6. (a) [1 pt] A parametrization of ℓ is

$$\ell\colon\ \mathbf{x}(t) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t+1 \\ t+2 \\ -t \end{bmatrix} \qquad t \in \mathbb{R}.$$

(b) [2 pt] Solve the equation $|\mathbf{x}(t)| = 1$:

$$\begin{aligned} |\mathbf{x}(t)|^2 &= 1^2 \\ (t+1)^2 + (t+2)^2 + (-t)^2 &= 1 \\ \left(t^2 + 2t + 1\right) + \left(t^2 + 4t + 4\right) + t^2 - 1 &= 0 \\ 3t^2 + 6t + 4 &= 0 \end{aligned}$$ **[1 pt]**

The discriminant is -12, so the quadratic equation has no real solutions. Conclusion: there are no points on ℓ with distance 1 to the origin. **[1 pt]**

7. (a) [2 pt] Let $z = \frac{1}{y}$, then

$$\begin{aligned} z' &= \frac{d}{dx}\left(\frac{1}{y}\right) \\ &= -\frac{1}{y^2}y' && \textbf{[0.5 pt]} \\ &= -\frac{1}{y^2}\left(4y^2 + y\right) \\ &= -4 - \frac{1}{y} && \textbf{[0.5 pt]} \\ &= -4 - z, && \textbf{[0.5 pt]} \end{aligned}$$

hence $\alpha = -1$ and $\beta = -4$. **[0.5 pt]**

(b) [2 pt] The differential equation $z' = -z - 4$ is linear, it is of the form $y' + P(x)y = Q(x)$ with $P(x) = 1$ and $Q(x) = -4$. The integrating factor is

$$v(x) = e^{\int P(x)\ \mathrm{d}x} = e^{\int 1\ \mathrm{d}x} = e^x. \qquad \textbf{[1 pt]}$$

Multiply the equation with $v(x)$: The general solution is

$$\begin{aligned} v(x)z' + v(x)z &= -4v(x) \\ e^x z' + \frac{d}{dx}\left(e^x\right)(x)\, z &= -4e^x \\ \frac{d}{dx}\left(e^x z(x)\right) &= -4e^x \\ e^x z(x) &= -4e^x + C \\ z(x) &= -4 + Ce^{-x}, \end{aligned}$$

hence $y(x) = \frac{1}{Ce^{-x} - 4}$. **[1 pt]**

Remark: For the derivation **[0.5 pt]** and for the answer **[0.5 pt]**

The escape equation $z' = -3z + 33$ is solved in the same way: the integrating factor is

$$v(x) = e^{\int 3\ \mathrm{d}x} = e^{3x}. \qquad \textbf{[1 pt]}$$

Multiply the equation with $v(x)$: the general solution is

$$\begin{aligned} v(x)z' + v(x)\cdot 3z &= 33v(x) \\ e^{3x}z' + \frac{d}{dx}\left(e^{3x}\right)\, z &= 33e^{3x} \\ \frac{d}{dx}\left(e^{3x}z(x)\right) &= 33e^{3x} \\ e^{3x}z(x) &= 11e^{3x} + C \\ z(x) &= 11 + Ce^{-3x}. && \textbf{[1 pt]} \end{aligned}$$

Remark: For the derivation **[0.5 pt]** and for the answer **[0.5 pt]**
Alternatively, you write the differential equation as

$$z' + z = -4$$

and solve the homogeneous equation

$$z' + z = 0$$

giving $z_c(x) = ce^{-x}$, **[1 pt]**, and find a particular solution $z_p(x) = -4$ **[0.5 pt]**. This leads to

$$z(x) = z_c(x) + z_p(x) = ce^{-x} - 4 \qquad \textbf{[0.5 pt]}$$

Remark: Similar for the escape equation $z' = -3z + 33$.