Discrete Mathematics for Computer Science, Solution/Correction standard, Sample Test; Part 2

1. Proof with Mathematical Induction (alternative form; Theorem 4.2).
   Basis step for $n = 1$, $n = 2$ and $n = 3$:

$$a_1 = 2 \leq \left(\frac{5}{2}\right)^1;\ a_2 = 6 \leq \left(\frac{25}{4}\right) = \left(\frac{5}{2}\right)^2;\ a_3 = 15 \leq \left(\frac{125}{8}\right) = \left(\frac{5}{2}\right)^3.$$

   Thus the statement is correct for $n = 1$, $n = 2$ and $n = 3$. **[1 pt]**
   Induction step:

   Let $k \geq 3$ and suppose that for all $1 \leq i \leq k$:

$$a_i \leq \left(\frac{5}{2}\right)^i \quad \text{(Induction Hypothesis: IH).}$$ **[1 pt]**

   We must show that IH implies:

$$a_{k+1} \leq \left(\frac{5}{2}\right)^{k+1}.$$ **[1 pt]**

   Well, we have

$$a_{k+1} = a_k + 2a_{k-1} + 4a_{k-2} \leq \left(\frac{5}{2}\right)^k + 2\left(\frac{5}{2}\right)^{k-1} + 4\left(\frac{5}{2}\right)^{k-2},$$

   where the inequality follows from the Induction Hypothesis. **[1 pt]**
   Now it suffices to show that

$$\left(\frac{5}{2}\right)^k + 2\left(\frac{5}{2}\right)^{k-1} + 4\left(\frac{5}{2}\right)^{k-2} \leq \left(\frac{5}{2}\right)^{k+1}.$$

   By dividing both sides of this inequality by $\left(\frac{5}{2}\right)^{k-2}$, we obtain the equivalent inequality

$$\left(\frac{5}{2}\right)^2 + 2\left(\frac{5}{2}\right)^1 + 4 \leq \left(\frac{5}{2}\right)^3.$$

   This inequality clearly holds, since

$$\frac{25}{4} + \frac{10}{2} + 4 = \frac{50 + 40 + 32}{8} = \frac{122}{8} \leq \frac{125}{8}.$$ **[2 pt]**

   This completes the proof by the principle of mathematical induction.
   (From the proof it must be crystal clear whàt is supposed **[1 pt]** and whàt must be proved **[1 pt]**. In case the induction hypothesis is not correctly formulated or the proof is not clearly written down: at most **1 pt** for the entire exercise)

2. (i) f is commutative, since for all $A, B \in \mathcal{P}(\mathcal{U})$ we have

$$f(A, B) = \overline{A \cup B} = \overline{B \cup A} = f(B, A).$$ **[2 pt]**

   (ii) f is *not* associative. E.g. take $\mathcal{U} = \{1\}; A = \{1\}; B = C = \varnothing$, then

$$f(f(A, B), C) = \overline{\overline{A \cup B} \cup C} = \{1\} \quad \text{but} \quad f(A, f(B, C)) = \overline{A \cup \overline{B \cup C}} = \varnothing.$$ **[2 pt]**

(iii) Suppose D is an identity of f. Then

$$f(D, A) = f(A, D) = A \quad \text{for all } A \in \mathcal{P}(\mathcal{U}).$$

So $\overline{D \cup A} = A$ for all $A \in \mathcal{P}(\mathcal{U})$. Taking $A = \varnothing$ yields $\overline{D} = \varnothing$. Therefore $D = \mathcal{U}$ is the only candidate for an identity. But $D = \mathcal{U}$ yields $\overline{\mathcal{U} \cup A} = A$ for all $A \in \mathcal{P}(\mathcal{U})$. Taking $A = \mathcal{U}$, we get $\mathcal{U} = \varnothing$, contradicting the assumption that $\mathcal{U}$ is nonempty.
Hence f does not have an identity. **[2 pt]**

3. Let $n = |A|$ and write $A = \{a_1, a_2, \ldots, a_n\}$.

(i) Proof of "$\Rightarrow$":
Suppose that R is transitive.
Recall that $M_{ij} \in \{0, 1\}$ for all i, j and that Boolean addition ($1 + 1 = 1$) is used to compute M^2. Hence, in order to prove that $M^2 \leq M$, it suffices to show that:

$$\text{if } M^2_{ij} = 1 \text{ for some } i, j \in \{1, 2, \ldots, n\}, \text{ then also } M_{ij} = 1.$$

So suppose $M^2_{ij} = 1$ for some $i, j \in \{1, \ldots, n\}$.
Then the dot product of the i^{th} row and j^{th} column of M is equal to 1: $\displaystyle\sum_{k=1}^{n} M_{ik} M_{kj} = 1.$
Therefore $\exists_{k \in \{1,2,\ldots,n\}} [M_{ik} M_{kj} = 1]$, so $\exists_{k \in \{1,2,\ldots,n\}} [M_{ik} = 1 \wedge M_{kj} = 1]$.
Hence, $\exists_{k \in \{1,2,\ldots,n\}} [a_i R a_k \wedge a_k R a_j]$, and therefore $a_i R a_j$, because R is transitive.
And so: $M_{ij} = 1$. **[3 pt]**

(ii) Proof of "$\Leftarrow$":
Suppose that $M^2 \leq M$.
In order to prove that R is transitive, let $a_i, a_k, a_j \in A$ be such that $a_i R a_k$ and $a_k R a_j$.
We will show that $a_i R a_j$.
Well: $a_i R a_k$ and $a_k R a_j$ imply that $M_{ik} = 1$ and $M_{kj} = 1$, and so the dot product of the i^{th} row and j^{th} column of M is equal to 1 (note: here Boolean addition is used).
So $M^2_{ij} = 1$, and therefore $M_{ij} = 1$ because $M^2 \leq M$.
Hence: $a_i R a_j$. **[3 pt]**