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Solution exam Linear Algebra on Friday March 26, 2021, 13.45 – 15.45 hours.

1.

> Consider three lines in $\mathbb{R}^3$:
>
> $$\ell_1 : \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R} \right\}$$
>
> $$\ell_2 : \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R} \right\}$$
>
> $$\ell_3 : \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R} \right\}$$
>
> Verify whether these three lines have a common intersection point and, if so, determine all these intersection points.

In order to solve this problem we reason as follows: Suppose $\mathbf{x} \in \mathbb{R}^3$ is a common point of intersection. Well, in that case $\mathbf{x}$ belongs (in particular) to the line ℓ_1. From the definition of ℓ_1 above we conclude that there exists a real number, let's say λ_1, such that we can write

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Now, since the point $\mathbf{x}$ also belongs to the lines ℓ_2 and ℓ_3, we conclude that there exist real numbers λ_2 and λ_3 such that we can write

$$\mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

We now have three different expressions representing *the same point* $\mathbf{x}$. Equating these expressions to each other, we obtain the following equations:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

To find this point of intersection (if it exists), we need to solve these equations for λ_1, λ_2 and λ_3.

We first compute intersection points of lines ℓ_1 and ℓ_2. We focus on the following equation:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

This yields the following linear system:

$$\begin{pmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

It is easily checked that this has a unique solution $\lambda_1 = -1$ and $\lambda_2 = 2$. Therefore the lines ℓ_1 and ℓ_2 intersect in exactly one point:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Next, we need to check whether this intersection point lies on ℓ_3. We find the condition

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

for some λ_3. Solving this equation we find $\lambda_3 = 1$. Hence our intersection point also lies on the line ℓ_3. Therefore the answer is:

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

2.

Find all possible α for which the volume of the parallelepiped with vertices $(\alpha, 0, 1)$, $(1, -1, 1)$, $(-1, 1, 2)$ and $(0, 0, 0)$ is equal to 6.

We should have:

$$\left| \det \begin{pmatrix} \alpha & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \right| = 6$$

This yields:

$$|-3\alpha| = 6$$

and we find $\alpha = 2$ or $\alpha = -2$.

3.

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

Determine the inverse of the matrix A and the inverse of the matrix A^{T}.

We have:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 2 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & -1 & 3 & 1 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -3 & -1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 1 & -2 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -3 & -1 & 2 \end{array}\right)$$

Therefore:

$$A^{-1} = \begin{pmatrix} 4 & 1 & -2 \\ -2 & 0 & 1 \\ -3 & -1 & 2 \end{pmatrix}$$

Next, we have:

$$(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}} = \begin{pmatrix} 4 & 1 & -2 \\ -2 & 0 & 1 \\ -3 & -1 & 2 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 4 & -2 & -3 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}.$$

4.

Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -2 & 1 & 1 & -1 \\ 2 & 0 & -1 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Given is that R is the row-reduced echelon form of the matrix A. In that case, a basis for $\operatorname{Col} A$ is given by:

a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$

c) $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

d) $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

Indicate which of the above four options are correct and which of these options are wrong.

First, we know that if we select the columns of A which have a pivot in the row-reduced echelon form, then we are guaranteed to get a bases for $\operatorname{Col} A$. So we can immediately deduced that option a) will be true. However, we know that bases are *not unique*, so we need to consider the other possibilities and decide which ones give a correct basis.

We can see from the row-reduced echelon form that the columns of A are not linearly independent. Indeed, we can see that $\operatorname{Null} A$ can be written as

$$\operatorname{Null} A = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}\right\}.$$

The vector in the nullspace gives explicitly a dependency in the columns of A, i.e. $\mathbf{a}_1 + \mathbf{a}_3 - \mathbf{a}_4 = \mathbf{0}$.

From this dependency we conclude that in order to form a basis for $\operatorname{Col} A$, we only need two out of the three vectors $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4$ to form a basis (plus the vector $\mathbf{a}_2$, which is linearly independent from the other columns).

To be more explicit, we can argue as follows: By definition, $\operatorname{Col} A$ is spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$. Given the dependency mentioned above, $\mathbf{a}_1$ can be expressed in terms of $\mathbf{a}_3$ and $\mathbf{a}_4$ and therefore

$$\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \operatorname{Span}\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$$

Similarly, $\mathbf{a}_3$ can be expressed in terms of $\mathbf{a}_1$ and $\mathbf{a}_4$, so

$$\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$$

and expressing $\mathbf{a}_4$ can be expressed in terms of $\mathbf{a}_1$ and $\mathbf{a}_3$ we get

$$\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$$

This implies that a), b) and d) are true. However, in c) we deleted $\mathbf{a}_2$ and according to R we cannot express $\mathbf{a}_2$ in terms of $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_4$ and therefore c) is false (equivalently, the vectors $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4$ are linearly dependent so they *do not* form a basis).

5.

Given is the matrix A

$$A = \begin{pmatrix}1 & \alpha & 0\\ 0 & \alpha+1 & 1\\ 0 & 0 & 3-\alpha\end{pmatrix}$$

where $\alpha \in \mathbb{R}$. Determine all $\alpha \in \mathbb{R}$ for which the matrix A is diagonalizable.

It is easy to verify that

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - \alpha - 1)(\lambda - 3 + \alpha)$$

and therefore the eigenvalues are 1, $\alpha + 1$ and $3 - \alpha$.

If we have three distinct eigenvalues then it is known that the matrix is diagonalizable. This is the case for $\alpha \neq 0, 1, 2$.

We still need to verify the three cases $\alpha = 0, 1, 2$. If $\alpha = 0$, we have:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

with eigenvalues 1 and 3. For the matrix to be diagonalizable, we need two independent eigenvalues associated to the double eigenvalue 1. It is easy to verify

$$E_1(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

so we do have two independent eigenvalues associated to the double eigenvalue 1 and therefore the matrix is diagonalizable.

If $\alpha = 1$, we have:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

with eigenvalues 1 and 2. For the matrix to be diagonalizable, we need two independent eigenvalues associated to the double eigenvalue 2. It is easy to verify

$$E_2(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

so we have only one independent eigenvalues associated to the double eigenvalue 2 and therefore the matrix is not diagonalizable.

If $\alpha = 2$, we have:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

with eigenvalues 1 and 3. For the matrix to be diagonalizable, we need two independent eigenvalues associated to the double eigenvalue 1. It is easy to verify

$$E_1(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

so we have only one independent eigenvalues associated to the double eigenvalue 1 and therefore the matrix is not diagonalizable.

Therefore, the matrix is diagonalizable for all $\alpha \in \mathbb{R}$ except for $\alpha = 1$ and $\alpha = 2$.

6.

$S : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which takes each point $(x_1, x_2) \in \mathbb{R}^2$ and rotates it first through 45 degrees (counterclockwise), then mirrors the result on the line $y = x$ and finally rotates it through 45 degrees (clockwise).
$T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which takes each point $(x_1, x_2) \in \mathbb{R}^2$ and rotates it first through 30 degrees (counterclockwise), then projects the result on the line $y = x$ and, finally, rotates it through 60 degrees (clockwise).

a) Determine the representation matrix of S.

We have:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Therefore the standard matrix is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A different approach is to compute the standard matrices for each of the steps defining the transformation S, and then multiplying them *in the correct order*. Let S_1 denote the first rotation (counterclockwise), S_2 denote the reflection, and S_3 denote the second rotation (clockwise). Then the standard matrices for these linear transformations are:

$$S_1 = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}.$$

Multiplying these matrices we get:

$$S = S_3\, S_2\, S_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b) Determine whether T is surjective (onto) and/or injective (one-to-one).

We note that $T = T_3 T_2 T_1$ is a composition of three linear transformations. The second one T_2 is a projection which is clearly neither one-to-one nor onto. On the other hand T_1 and T_3 are, as rotations, obviously invertible. But this implies that T is neither one-to-one nor onto. For instance.

$$T_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

yields

$$T\left(T_1^{-1}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 0$$

which shows T is not one-to-one. As a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ this yields that T is not onto either.

The same conclusion can also be reached by thinking about the determinants. The determinant of

$$T_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

is cleary zero (from the above expression or from the fact that it is a projection onto a line). This means that also the determinant of T is zero, and thus T is neither one-to-one nor onto.

7.

> a) Show that if a matrix T has eigenvalue 1 then the matrix T^2 also has an eigenvalue in 1.

If T has eigenvalue 1 then there exists a vector $\mathbf{x} \neq 0$ such that:

$$T\mathbf{x} = \mathbf{x}$$

But this implies

$$T^2\mathbf{x} = T\left(T\mathbf{x}\right) = T\mathbf{x} = \mathbf{x}$$

and therefore $\mathbf{x}$ is also an eigenvector of T^2 with eigenvalues 1 which completes the proof.

> b) Verify whether if a matrix T is such that matrix T^2 has eigenvalue 1 then the matrix T itself also has an eigenvalue in 1. If this is true prove it; if this is not true then give a counterexample.

No this is not true. For instance, if

$$T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

clearly has eigenvalue 1 but the matrix T has only eigenvalues in -1.

8.

> Consider the matrices A and B given by:
>
> $$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & -2 \end{pmatrix}.$$
>
> It is given that $\text{Null}\, A = \{\mathbf{0}\}$ and $\text{Col}\, B = \mathbb{R}^3$.
>
> a) Determine $\text{Null}\, AB$

The easiest, and safest way to solve the problem is using some theorey to reason like this:

Because $\operatorname{Null} A = \{\mathbf{0}\}$ we know that $AB\mathbf{x} = \mathbf{0}$ implies $B\mathbf{x} = \mathbf{0}$. Therefore $\operatorname{Null} AB = \operatorname{Null} B$. We have:

$$B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -3 \\ 0 & -1 & 1 & -2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & -3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

We find:

$$\operatorname{Null} AB = \operatorname{Null} B = \operatorname{Span} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

Of course, it is also possible to explicitly compute AB and work from there. However, this mehtod is slower, requires more computations, and is much more prone to mistakes! For the sake of completeness, we show here the product AB and its row-reduced echelon form:

$$AB = \begin{pmatrix} 3 & 1 & 1 & 3 \\ 4 & -1 & 3 & 0 \\ 1 & 2 & -1 & 4 \\ -3 & -2 & 0 & -5 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

b) Determine $\operatorname{Col} AB$

Because $\operatorname{Col} B = \mathbb{R}^3$, we have that $\operatorname{Col} AB = \operatorname{Col} A$. Moreover, since $\operatorname{Null} A = \{0\}$ the columns of A are independent. We find:

$$\operatorname{Col} AB = \operatorname{Col} A = \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

9.

Given are two bases of $\mathbb{R}^3$:

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{T} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \right\}$$

For a vector $\mathbf{x} \in \mathbb{R}^3$ it is given that

$$[\mathbf{x}]_{\mathcal{S}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Determine $[\mathbf{x}]_{\mathcal{T}}$.

Given $[\mathbf{x}]_{\mathcal{S}}$, we can compute $\mathbf{x}$:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Next we have to express $\mathbf{x}$ in terms of the basis vectors of $\mathcal{T}$:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = t_1 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t_3 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

This yields

$$[\mathbf{x}]_{\mathcal{T}} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$