# Test Probability Theory, 14 June 2019

module Data & Information

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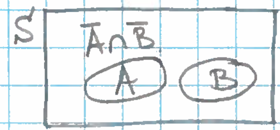
**1** a) $P(A \cup B) = 1 - P(\bar{A} \cap \bar{B}) = 1 - 0.2 = 0.8$




b) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0}{0.6} = 0$

because A and B are mutually exclusive $(P(A \cap B) = 0)$.

c) $P(A \cup B) = P(A) + P(B) = 0.8$, so $P(A) = 0.8 - 0.6 = 0.2$.

$0 = P(AB) \neq P(A) \cdot P(B) = 0.2 \cdot 0.6$

↳ no, A and B are not independent.

**2** a) X: number of tagged animals in the second sample.

Sampling without replacement

→ hypergeometric

| | Tagged | not | Total |
|---|---|---|---|
| | 5 | 20 | 25 |
| | ↓ | ↓ | ↓ |
| | 1 | 9 | 10 |
| or | 0 | 10 | |

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \frac{\binom{5}{0}\binom{20}{10}}{\binom{25}{10}} + \frac{\binom{5}{1}\binom{20}{9}}{\binom{25}{10}}$$

$$= \frac{187{,}756 + 839{,}800}{3{,}268{,}760} \approx 31.4\ \%$$

b) $\text{var}(X) = n \cdot \frac{R}{N} \cdot \left(1 - \frac{R}{N}\right) \cdot \frac{N-n}{N-1}$ (formula sheet)

$$= 10 \cdot \frac{5}{25} \cdot \left(1 - \frac{5}{25}\right) \cdot \frac{15}{24} = 10 \cdot \frac{1}{5} \cdot \frac{4}{5} \cdot \frac{5}{8} = 1$$

$\sigma_X = \sqrt{1} = 1$

**3** a)

| $i \backslash j$ | $-1$ | $0$ | $1$ | $P(X=i)$ ← row-sum |
|---|---|---|---|---|
| $-1$ | $1/4$ | $1/6$ | $1/12$ | $1/2$ |
| $0$ | $1/6$ | $1/9$ | $1/18$ | $1/3$ |
| $1$ | $1/12$ | $1/18$ | $1/36$ | $1/6$ |
| | $(P(Y=j) = P(X=j))$ | | | |

$E(X) = \sum_x x \cdot P(X=x) = (-1) \cdot \frac{1}{2} + 1 \cdot \frac{1}{6} = -\frac{1}{3}$

$E(X^2) = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{6} = \frac{2}{3}$

$\text{var}(X) = E(X^2) - (E(X))^2 = \frac{2}{3} - \left(-\frac{1}{3}\right)^2 = \frac{5}{9}$.

b) By symmetry: $E(Y) = E(X)$ and $\text{var}(Y) = \text{var}(X)$.

$E(XY) = \sum_i \sum_j i j \cdot P(X=i \text{ and } Y=j) = 1 \cdot \frac{1}{4} - 1 \cdot \frac{1}{12} - 1 \cdot \frac{1}{12} + 1 \cdot \frac{1}{36} = \frac{1}{9}$.

(3b cont.)

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{9} - \left(-\frac{1}{3}\right)^2 = 0$$

so, $$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y} = 0.$$

c) Yes. For all $i$ and $j$ $P(X=i \text{ and } Y=j) = P(X=i)\cdot P(Y=j)$.

For example, $\frac{1}{4} = P(X=-1 \text{ and } Y=-1) = P(X=-1)\cdot P(Y=-1) = \frac{1}{2}\cdot\frac{1}{2}$.

d) $$P(X>Y) = P(X=0 \text{ and } Y=-1) + P(X=1 \text{ and } Y=-1) + P(X=1 \text{ and } Y=0)$$
$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{18} = \frac{11}{36}.$$

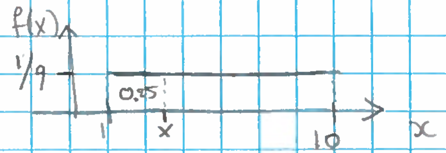
e) $$P(Y=-1 \mid X=0) = \frac{P(X=0 \text{ and } Y=-1)}{P(X=0)} = \frac{1/6}{1/3} = \frac{1}{2}, \text{ and so on.}$$

| $y$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $P(Y=y \mid X=0)$ | $1/2$ | $1/3$ | $1/6$ |

Or by independence: $P(Y=y \mid X=0) = P(Y=y)$.

$E(Y \mid X=0) = E(Y) = -\frac{1}{3}$.

4. a)




$$P(X \le x) = 0.25, \text{ so } \overbrace{(x-1)\cdot\frac{1}{9}}^{\text{area}} = 0.25 \Rightarrow x = 3.25.$$

b) $$F_X(x) = P(X \le x) = \begin{cases} 0, & x < 1 \\ (x-1)/9, & 1 \le x \le 10 \quad \text{(see (a))} \\ 1, & x > 10 \end{cases}$$

c) $$F_Y(y) = P(Y \le y) = P\left(\frac{10}{X} \le y\right) = P\left(X \ge \frac{10}{y}\right) = 1 - F_X\left(\frac{10}{y}\right)$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(1 - F_X\left(\frac{10}{y}\right)\right) = -f_X\left(\frac{10}{y}\right)\cdot\left(-\frac{10}{y^2}\right)$$

$$= \begin{cases} \frac{1}{9}\cdot\frac{10}{y^2}, & 1 \le 10/y \le 1 \Leftrightarrow 1 \le y \le 10 \leftarrow \text{range} \\ 0, & \text{otherwise.} \end{cases}$$

5. a) $X$: IQ of person in the group, $X \sim N(100, 16^2)$

$$P(115 < X < 140) = P\left(\frac{115-100}{16} < Z < \frac{140-100}{16}\right)$$

$$\approx P(0.94 < Z < 2.50) = \Phi(2.50) - \Phi(0.94)$$

$$= 0.9938 - 0.8264 = 0.1674.$$

b) By the empirical rule: $(\mu - \sigma, \mu + \sigma) = (84, 116)$.

c) (assume independence of IQs)

$$\overline{X}_{64} \sim N\left(100, \frac{16^2}{64}\right) = N(100, 4).$$

$$P(\overline{X}_{64} \leq 103) = P\left(Z \leq \frac{103-100}{\sqrt{4}}\right) = \Phi(1.50) = 0.9332.$$

6 a) (formula sheet) $E(X_i) = 1/\lambda = 10 \Rightarrow \lambda = 1/10$.

$$P(X_1 > 12) = e^{-\lambda \cdot 12} = e^{-6/5} \approx 30.1\%$$

$$P(X_1 > 20 \mid X_1 > 8) = P(X_1 > 12)$$ (lack of memory property)

$$\approx 30.1\%.$$

b) $S_{10}$ has an Erlang distribution with parameters $n = 10$ and $\lambda = \frac{1}{10}$. (or mention density, formula sheet)

$$\left.\begin{aligned} E(S_{10}) &= n/\lambda = 100 \\ \mathrm{var}(S_{10}) &= n/\lambda^2 = 1000 \end{aligned}\right\} \text{formula sheet}$$