LINEAR ALGEBRA Date : March 26, 2021
Time : 13.45 – 15.45 hrs

First read these instructions carefully:

This test contains 9 exercises. The complete solutions of Exercises 5, 6, 7, 8 and 9 must be accurately written down on a separate sheet, including calculations and argumentation.
For the other exercises you are only required to fill in the final answers on the answer sheet at the end of this test. You must hand in this answer sheet as well as your hand written solutions to Exercises 5, 6, 7, 8 and 9.
The use of electronic devices is not allowed.

1. *Fill in your final answers to this exercise on the supplied answer sheet!*

   Consider three lines in $\mathbb{R}^3$:

$$\ell_1 : \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R} \right\}$$

$$\ell_2 : \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R} \right\}$$

$$\ell_3 : \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R} \right\}$$

   Verify whether these three lines have a common intersection point and, if so, determine all these intersection points.

2. *Fill in your final answers to this exercise on the supplied answer sheet!*

   Find all possible α for which the volume of the parallelepiped with vertices $(\alpha, 0, 1)$, $(1, -1, 1)$, $(-1, 1, 2)$ and $(0, 0, 0)$ is equal to 6.

3. *Fill in your final answers to this exercise on the supplied answer sheet!*

   Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

   Determine the inverse of the matrix A and the inverse of the matrix A^{T}.

4. *Fill in your final answers to this exercise on the supplied answer sheet!*

   Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -2 & 1 & 1 & -1 \\ 2 & 0 & -1 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Given is that R is the row-reduced echelon form of the matrix A. In that case, a basis for Col A is given by:

   a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ c) $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

   b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ d) $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

   Indicate which of the above four options are correct and which of these options are wrong.

5. *Use separate sheet and include clear argumentation and calculation!*

   Given is the matrix A

$$A = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & \alpha+1 & 1 \\ 0 & 0 & 3-\alpha \end{pmatrix}$$

   where $\alpha \in \mathbb{R}$. Determine all $\alpha \in \mathbb{R}$ for which the matrix A is diagonalizable.

6. *Use separate sheet and include clear argumentation and calculation!*

   $S : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which takes each point $(x_1, x_2) \in \mathbb{R}^2$ and rotates it first through 45 degrees (counterclockwise), then mirrors the result on the line $y = x$ and finally rotates it through 45 degrees (clockwise).

   $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which takes each point $(x_1, x_2) \in \mathbb{R}^2$ and rotates it first through 30 degrees (counterclockwise), then projects the result on the line $y = x$ and, finally, rotates it through 60 degrees (clockwise).

   a) Determine the representation matrix of S.
   b) Determine whether T is surjective (onto) and/or injective (one-to-one).

7. *Use separate sheet and include clear argumentation and calculation!*

   a) Show that if a matrix T has eigenvalue 1 then the matrix T^2 also has an eigenvalue in 1.
   b) Verify whether if a matrix T is such that matrix T^2 has eigenvalue 1 then the matrix T itself also has an eigenvalue in 1. If this is true prove it; if this is not true then give a counterexample.

8. *Use separate sheet and include clear argumentation and calculation!*

   Consider the matrices A and B given by:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & -2 \end{pmatrix}.$$

   It is given that $\operatorname{Null} A = \{\mathbf{0}\}$ and $\operatorname{Col} B = \mathbb{R}^3$.

   a) Determine $\operatorname{Null} AB$

   b) Determine $\operatorname{Col} AB$

9. *Use separate sheet and include clear argumentation and calculation!*

   Given are two bases of $\mathbb{R}^3$:

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{T} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \right\}$$

   For a vector $\mathbf{x} \in \mathbb{R}^3$ it is given that

$$[\mathbf{x}]_{\mathcal{S}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

   Determine $[\mathbf{x}]_{\mathcal{T}}$.

For the exercises the following number of points can be obtained:

Exercise	1.	3 points	Exercise	2.	3 points	Exercise	3.	3 points
Exercise	4.	3 points	Exercise	5.	4 points	Exercise	6a.	3 points
Exercise	6b.	3 points	Exercise	7a	2 points	Exercise	7b.	3 points
Exercise	8a.	3 points	Exercise	8b.	3 points	Exercise	9.	3 points

The grade is determined by dividing the total number of points by 4 and adding 1.