Exam for Limits to Computing (201300042)
Thursday October 31, 08:45 – 12:15

(No books/printouts/etc. may be used during this exam. Explain your answers.)

1. For each of the following decision problems, which take a standard representation $R(M)$ of a Turing machine M with tape alphabet $\{0, 1, B\}$ as input, argue whether it is decidable.

   (a) Does M accept the input string $R(M)$?

   (b) Does M accept the empty input λ while using only the first 100 tape positions during the computation?

   (c) Do all strings in $L(M)$ have length at most 10?

   (d) Is $\overline{L(M)}$ recursively enumerable?

2. Let M be a Turing Machine with space complexity $s(n)$. For any (real) constant $c > 0$, denote by $s^c(n)$ the function $n \rightarrow \max\{n + 2, \lceil c \cdot s(n) \rceil\}$.

   (a) For any constant $c > 0$, show how to construct a two-tape Turing Machine M' with $L(M) = L(M')$ which is $s^c(n)$-space bounded.

   (b) For any constant $c > 0$, show how to construct a (one tape) Turing Machine M' with $L(M) = L(M')$ which is $s^c(n)$-space bounded.

   (Only the construction of the respective Turing Machines needs to be outlined. For constructions that were presented in the lecture, it is not required to repeat precise implementation details or correctness proofs.)

3. (a) An *independent set* of a graph G is a set of vertices $S \subseteq V(G)$ such that there is no edge of G with two end vertices in S. Prove that the following problem is $\mathcal{NP}$-complete, using a reduction from the Vertex Cover Problem:

   Independent Set:
   INSTANCE: A graph G and integer k.
   QUESTION: Does G admit an independent set S with $|S| \geq k$?

   (b) Let G be a graph with vertex weights $w : V(G) \rightarrow \mathbb{N}$. The *weight of a subgraph H of G* is defined as $\sum_{v \in V(H)} w(v)$. Prove that the following problem is $\mathcal{NP}$-complete.

   Weighted Connected Subgraph:
   INSTANCE: A *tree* T with vertex weights $w : V(T) \rightarrow \mathbb{N}$, and integer M.
   QUESTION: Does there exist a *connected* subgraph H of T with weight exactly M?

4. Let LIN denote the set of languages that can be recognized in time $O(n)$ by a multitape Turing Machine, and let NLIN denote the set of languages that can be recognized in time $O(n)$ by a multitape *nondeterministic* Turing Machine. (As usual, n denotes the size of the input.)

   (a) Prove *one of* the following two statements:

   - co-LIN = LIN.
   - co-NLIN = NLIN.

   (b) For two languages L_1 and L_2, denote

   $$L_1 \circ L_2 = \{vw \mid v \in L_1 \wedge w \in L_2\}.$$

   Prove *one of* the following two statements:

   - If $L_1 \in$ LIN and $L_2 \in$ LIN then $L_1 \circ L_2 \in$ LIN.
   - If $L_1 \in$ NLIN and $L_2 \in$ NLIN then $L_1 \circ L_2 \in$ NLIN.

5. (a) Prove that $\mathcal{P}$-SPACE = co-$\mathcal{P}$-SPACE.

   (b) Prove that if there exists a language $L \in \mathcal{NP}$ that is $\mathcal{P}$-SPACE-complete, then every language in $\mathcal{P}$-SPACE can be recognized by a polynomial time nondeterministic Turing Machine. What is the degree of the resulting polynomial that bounds the time complexity?

   (c) Prove that if there exists a language $L \in$ co-$\mathcal{NP}$ that is $\mathcal{P}$-SPACE-complete, then every language in $\mathcal{P}$-SPACE can be recognized by a polynomial time nondeterministic Turing Machine.

Points:

1.(a): 1	2.(a): 2	3.(a): 2	4.(a): 2	5.(a): 1
(b): 2	(b): 2	(b): 3	(b): 2	(b): 2
(c): 1				(c): 2
(d): 2				

Total: $24 + 3 = 27$ points.